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ON THE SUMMATION OF INFINITE SERIES OF LEGENDRE'S FUNCTIONS

(*Second Paper*) *

BY

GANESH PRASAD

(*Calcutta University*)

I. $P_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}.P_{+\frac{1}{2}}(\cosh\sigma).\,r+\frac{1.3}{2.4}\cdot P_{\frac{3}{2}}(\cosh\sigma).\,r^2+\ldots\ldots$to infinity

$$=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\{\cosh\sigma-r+\sinh\sigma\cos\phi\}^{\frac{1}{2}}}=\frac{2K}{\pi\sqrt{e^{\sigma}-r}}$$

the modulus corresponding to K being $\sqrt{\dfrac{2\sinh\sigma}{e^{\sigma}-r}}$ and r being $<e^{-\sigma}$.

Proof :

It is known[†] that

$$P_{m-\frac{1}{2}}(\cosh\sigma)=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\{\cosh\sigma+\sinh\sigma\cos\phi\}^{m+\frac{1}{2}}}$$

Therefore the series has the sum

$$\frac{1}{\pi}\int_0^{\pi}d\phi\left\{\sum_{m=0}^{\infty}\frac{r^m}{D^{m+\frac{1}{2}}}\cdot\frac{1.3.5\ldots(2m-1)}{2.4.6\ldots2m}\right\}$$

(D standing for $\cosh\sigma+\sinh\sigma\cos\phi$)

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[*] This paper was received on 27. 6.'31 and read on 12. 7.'31. The first paper appeared in Vol. XXII, 1930.

[†] See Hobson's paper "On a type of spherical harmonics of unrestricted degree, order and argument" (*Phil. Trans., A.*, Vol. 187, 1896, p. 524).

$$=\frac{1}{\pi}\int_0^\pi d\phi\Big/\sqrt{D-r}=\frac{1}{\pi}\int_0^\pi\frac{d\phi}{\{\cosh\sigma-r+\sinh\sigma\cos\phi\}^{\frac{1}{2}}}$$

$$=\frac{1}{\pi}\int_0^\pi\frac{d\phi}{\{\cosh\sigma-r+\sinh\sigma-2\sinh\sigma.\sin^2\frac{\phi}{2}\}^{\frac{1}{2}}}$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\{e^{\sigma}-r-2\sinh\sigma.\sin^2\theta\}^{\frac{1}{2}}}\ \left(\text{putting } \theta=\frac{\phi}{2}\right)$$

$$=\frac{2}{\pi\sqrt{e^{\sigma}-r}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\left(k^2 \text{ standing for } \frac{2\sinh\sigma}{e^{\sigma}-r}\right)$$

$$=\frac{2K}{\pi\sqrt{e^{\sigma}-r}}$$

II. $K_1(\cosh\psi)-\frac{1}{3}K_3(\cosh\psi)+\frac{1}{5}K_5(\cosh\psi)-\ldots$ to inf.

$$=\frac{1}{2}k^{\frac{1}{2}}F(k'),\ 0<\psi<\frac{\pi}{2},\ \text{where } k=e^{\psi}.$$

Proof :

It is known * that

$$K_p(\cosh\psi)=\frac{2}{\pi}\int_0^{\psi}\frac{\cos p\phi\,d\phi}{\{2(\cosh\psi-\cosh\phi)\}^{\frac{1}{2}}}.$$

Therefore the series in question has the sum

$$\frac{2}{\pi}\int_0^{\psi}\frac{d\phi}{\{2\cosh\psi-\cosh\phi)\}^{\frac{1}{2}}}\left\{\sum_{p=1}^{\infty}\frac{(-1)^{p-1}\cos(2p-1)\phi}{2p-1}\right\}\quad\ldots(1)$$

Now the cosine series within the crooked brackets equals $\frac{\pi}{4}$ or $-\frac{\pi}{4}$

Ibid, p. 580.

according as cos ϕ is + or − ; therefore, assuming that $0<\psi<\frac{\pi}{2}$, we have the sum equal to $\frac{\pi}{4}$.

Therefore (1) equals

$$\frac{1}{2}\int_0^{\psi}\frac{d\phi}{\{2(\cosh\psi-\cosh\phi)\}^{\frac{1}{2}}}=\frac{1}{2}\,k^{\frac{1}{2}}F(k'),$$

k standing for e^{ψ} .

For values of ψ outside the interval $(0, \frac{\pi}{2})$, the series can be easily summed up by a procedure similar to the above.

III. $Q_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,Q_{\frac{1}{2}}(\cosh\sigma).\,r+\frac{1.3}{2.4}\,Q_{\frac{3}{2}}(\cosh\sigma).r^2+\ldots$ to inf.

$=\frac{2K}{\sqrt{e^{\sigma}-r}}$, where the modulus of K is $\sqrt{\left\{\frac{e^{-\sigma}-r}{e^{\sigma}-r}\right\}}$ and $r<e^{\sigma}$.

Proof :

It is known * that

$$Q_{n-\frac{1}{2}}(\cosh\sigma)=\int_0^{\infty}\frac{d\theta}{(\cosh\sigma+\sinh\sigma\cosh\theta)^{n+\frac{1}{2}}}.$$

Therefore, proceeding as in the case of I, we have the sum of the series equal to

$$\int_0^{\infty}\frac{d\theta}{(\cosh\sigma-r+\sinh\sigma\cosh\theta)^{\frac{1}{2}}}$$

$$=2\int_0^{\infty}\frac{dt}{(\cosh\sigma-r+\sinh\sigma.\cosh 2t)^{\frac{1}{2}}}\text{, where } \theta=2t.$$

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* *Ibid*, p. 524.

Now the above

$$=2\int_0^{\infty}\frac{dt}{(2\sinh\sigma\cosh^2 t-r+e^{-\sigma})^{\frac{1}{2}}}$$

$$=2\int_0^{\frac{\pi}{2}}\frac{\sec\phi d\phi}{(2\sinh\sigma.\sec^2\phi-r+e^{-\sigma})^{\frac{1}{2}}}\text{ (where }\cosh t=$$

$$=2\int_0^{\frac{\pi}{2}}\frac{d\phi}{\{2\sinh\sigma+e^{-\sigma}-r-(e^{-\sigma}-r)\sin^2\phi\}^{\frac{1}{2}}}$$

$$=\frac{2}{\sqrt{2\sinh\sigma+e^{-\sigma}-r}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\{1-k^2\sin^2\phi\}^{\frac{1}{2}}}\left(\text{ where }k^2\right.$$

$$=\frac{2K}{\sqrt{e^{\sigma}-r}}\text{, where the modulus of K is }\sqrt{\left\{\frac{e^{-\sigma}-r}{e^{\sigma}-r}\right.}$$

IV. $Q_{\frac{1}{2}}(\cosh\psi)-\frac{1}{3}Q_{\frac{5}{2}}(\cosh\psi)+\frac{1}{5}Q_{\frac{9}{2}}(\cosh\psi)-\ldots$ to in

$$=\frac{\pi}{4}\frac{1}{\sinh\frac{\psi}{2}}F\left(\frac{\pi}{4},\frac{i}{\sinh\frac{\psi}{2}}\right)-\frac{\pi}{4}\frac{1}{\cosh\frac{\psi}{2}}F\left(\frac{\pi}{4},\frac{1}{\cosh\frac{\psi}{2}}\right.$$

Proof :

It is known * that

$$Q_{m-\frac{1}{2}}(\cosh\psi)=\int_0^{\pi}\frac{\cos m\theta\, d\theta}{\sqrt{2(\cosh\psi-\cos\theta)}}.$$

Therefore, proceeding as in the case of II, the sum of is easily found to be equal to

$$\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{2(\cosh\psi-\cos\theta)}}-\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{2(\cosh\psi+\cos\theta)}}$$

---

* Hobson, *l.c.*, p. 524.

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\frac{d\phi}{\sqrt{2(\cosh\psi-\cos 2\phi)}}-\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\frac{d\phi}{\sqrt{2(\cosh\psi+\cos 2\phi)}}$$

(putting $\theta=2\phi$)

$$=\frac{\pi}{4}\int_0^{\frac{\pi}{4}}\frac{d\phi}{\sqrt{(\sinh^2\frac{\psi}{2}+\sin^2\phi)}}-\frac{\pi}{4}\int_0^{\frac{\pi}{4}}\frac{d\phi}{\sqrt{(\cosh^2\frac{\psi}{2}-\sin^2\phi)}}$$

$$=\frac{\pi}{4}\cdot\frac{1}{\sinh\frac{\psi}{2}}\,F\left(\frac{\pi}{4},\frac{i}{\sinh\frac{\psi}{2}}\right)-\frac{\pi}{4\cosh\frac{\psi}{2}}\,F\left(\frac{\pi}{4},\frac{1}{\cosh\frac{\psi}{2}}\right).$$

V. $Q_{-\frac{1}{2}}(\cosh\psi)+Q_{\frac{1}{2}}(\cosh\psi)+Q_{\frac{3}{2}}(\cosh\psi)+\ldots$ to inf.

$$=\frac{1}{2}\,F\left(\frac{\pi-\psi}{2},\ \cosh\frac{\psi}{2}\right)+\frac{1}{2}\,\wp^{-1}(\gamma, g_2, g_3),$$

where $\gamma=\sec^2\frac{\psi}{2}-\frac{k^2+2}{3}$, k standing for $\cosh\frac{\psi}{2}$, and g_2, g_3 are quantities expressible in terms of k.

Proof :

It is known[*] that

$$Q_{n-\frac{1}{2}}(\cosh\psi)=\int_\psi^\infty\frac{e^{-nu}du}{(2\cosh u-2\cosh\psi)^{\frac{1}{2}}}.$$

Therefore the sum of the series in question is

$$\int_\psi^\infty\frac{1}{(2\cosh u-2\cosh\psi)^{\frac{1}{2}}}\left\{\sum_0^\infty e^{-nu}\right\}du$$

$$=\int_\psi^\infty\frac{e^u}{e^u-1}\cdot\frac{du}{(2\cosh u-2\cosh\psi)^{\frac{1}{2}}}$$

$$=2\int_{\frac{\psi}{2}}^\infty\frac{dt\,e^{2t}}{e^{2t}-1}\quad\frac{1}{(2\cosh 2t-2\cosh\psi)^{\frac{1}{2}}}. \tag{1}$$

(putting $u=2t$)

[*] Hobson, *l.c.*, p. 519.

Put $\cosh t = \sec\phi$, then $\sinh t = \tan\phi$, $dt = \sec\phi\, d\phi$. Therefore (1) is equal to

$$\int_{\frac{\psi}{2}}^{\frac{\pi}{2}} \frac{\sec\phi + \tan\phi}{\tan\phi} \cdot \frac{\sec\phi\, d\phi}{\left(4\sec^2\phi - 4\cosh^2\frac{\psi}{2}\right)^{\frac{1}{2}}}$$

$$= \tfrac{1}{2}\int_{\frac{\psi}{2}}^{\frac{\pi}{2}} \left(1 + \frac{1}{\sin\phi}\right) \cdot \frac{d\phi}{\left\{1 - \cosh^2\frac{\psi}{2} \,.\, \cos^2\phi\right\}^{\frac{1}{2}}}$$

$$= \tfrac{1}{2}\int_{0}^{\frac{\pi}{2}-\frac{\psi}{2}} \left(1 + \frac{1}{\cos v}\right) \frac{dv}{\left\{1 - \cosh^2\frac{\psi}{2}\ \sin^2 v\right\}^{\frac{1}{2}}}$$

(putting $v = \frac{\pi}{2} - \phi$)

$$= \tfrac{1}{2}\, F\left(\frac{\pi-\psi}{2}, \cosh\frac{\psi}{2}\right) + \tfrac{1}{2}\int_{0}^{\frac{\pi-\psi}{2}} \frac{dv}{\cos v\,(1 - k^2 \sin^2 v)^{\frac{1}{2}}}.$$

The second term in the above

$$= \tfrac{1}{2}\int_{0}^{\cos\frac{\psi}{2}} dx/\{1 - x^2)(1 - k^2x^2)^{\frac{1}{2}}\} \quad \text{(putting } \sin v = x)$$

$$= \tfrac{1}{2}\int_{\sec^2\frac{\psi}{2}}^{\infty} \frac{dy}{\sqrt{4(y-1)^2(y-k^2)}} \qquad \text{(putting } x^2 = \frac{1}{y})$$

$$= \tfrac{1}{2}\int_{\gamma}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} = \tfrac{1}{2}\wp^{-1}(\gamma, g_2, g_3)$$

(putting $z = y - \frac{k^2+2}{3}$)

where $\gamma = \sec^2\frac{\psi}{2} - \frac{k^2+2}{3}$ and g_2, g_3 are suitable quantities which are expressible in terms of k.

VI. $P_{-\frac{3}{4}}(\cosh\sigma)+\frac{1}{4}P_{\frac{1}{4}}(\cosh\sigma)r+\frac{1.5}{4.8}P_{\frac{5}{4}}(\cosh\sigma)r^2+\ldots$ to inf. is

$$\frac{4K}{\pi(e^{\sigma}-r)^{\frac{1}{4}}\sqrt{2+2k'}},\text{ where the modulus of K is }\frac{1-\sqrt{k'}}{\sqrt{2+2k'}},$$

$$k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}\text{ and }r<e^{-\sigma}.$$

Proof :

Proceeding as in the case of I, the sum of the series is found to be

$$\frac{1}{\pi}\int_0^{\pi}d\phi\left\{\sum_{m=0}^{\infty}\frac{r^m}{D^{m+\frac{1}{4}}}\frac{1.5.9\ldots(4m-3)}{4.8.12\ldots4m}\right\}$$

$$=\frac{1}{\pi}\int_0^{\pi}d\phi/(D-r)^{\frac{1}{4}}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\{e^{\sigma}-r-2\sinh\sigma\sin^2\theta\}^{\frac{1}{4}}}$$

$$=\frac{2}{\pi(e^{\sigma}-r)^{\frac{1}{4}}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{4}}}$$

(k^2 standing for $\frac{2\sinh\sigma}{e^{\sigma}-r}$)

$$=\frac{4K}{\pi(e^{\sigma}-r)^{\frac{1}{4}}\sqrt{2+2k'}}$$

(See Greenhill's *Elliptic Functions*, p. 164.)

VII. $(\log 2).\ Q_{-\frac{1}{2}}(\cosh\sigma)+Q_{\frac{3}{2}}(\cosh\sigma)+\frac{1}{2}Q_{\frac{7}{2}}(\cosh\sigma)$
$+\frac{1}{3}Q_{\frac{11}{2}}(\cosh\sigma)+\ldots$ to inf. $=\Phi$,

where Φ is a function expressible in terms of elliptic integrals.

Proof :

It is known that

$$Q_{m-\frac{1}{2}}(\cosh\sigma)=\int_0^{\pi}\frac{(-1)^m\cos m\theta d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}$$

Therefore the sum of the series is equal to

$$\int_0^{\pi}\frac{d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}\left\{\log 2+\sum_1^{\infty}\frac{\cos 2k\theta}{k}\right\}. \qquad (1)$$

But $\log 2+\sum_{1}^{\infty}\frac{\cos 2k\theta}{k}=-\log\sin\theta$. Hence (1) is equal to

$$-\frac{1}{2}\int_0^{\pi}\frac{\log\sin\theta d\theta}{\sqrt{\cosh^2\frac{\sigma}{2}-\sin^2\frac{\theta}{2}}}$$

$$=-\int_0^{\frac{\pi}{2}}\frac{\log\sin 2\phi\, d\phi}{\sqrt{\cosh^2\frac{\sigma}{2}-\sin^2\phi}}$$

(putting $\theta=2\phi$)

$$=-\frac{1}{\cosh\frac{\sigma}{2}}\int_0^{\frac{\pi}{2}}\frac{(\log 2+\log\sin\phi+\log\cos\phi)}{(1-k^2\sin^2\phi)^{\frac{1}{2}}}d\phi$$

$\left(\text{where } k^2=\operatorname{sech}^2\frac{\sigma}{2}\right)$

$$=-\frac{1}{\cosh\frac{\sigma}{2}}\left\{\mathrm{K}\log 2+\tfrac{1}{2}\mathrm{K}\log\frac{1}{k}-\frac{\pi}{4}\mathrm{K}'\right.$$

$$\left.+\frac{1}{k'}\left(\tfrac{1}{2}\overline{\mathrm{K}}\log\frac{1}{\bar{k}}-\frac{\pi}{4}\overline{\mathrm{K}'}\right)\right\}$$

where $\overline{\mathrm{K}}$, $\overline{\mathrm{K}}'$ are the same functions of $\bar{k}=\frac{ik}{k'}$, as K and K′ are of k.

VIII. $Q_{-\frac{1}{4}}(\cosh\sigma)+\frac{1}{4}Q_{\frac{1}{4}}(\cosh\sigma)r+\frac{1.5}{4.8}Q_{\frac{5}{4}}(\cosh\sigma).r^2$

$+\ldots$to infinity is easily summable for $r<e^{\sigma}$.

Proof :

Proceeding as in the case of III, the sum of the series is found to be

$$\int_0^{\infty}\frac{d\theta}{(\cosh\sigma-r+\sinh\sigma\cosh\theta)^{\frac{1}{2}}}$$

which is seen to be equal to

$$\frac{2}{(e^{\sigma}-r)^{\frac{1}{4}}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\cos\phi}\,(1-k^2\sin^2\phi)^{\frac{1}{4}}}=\Phi_1$$

$$\left(k^2 \text{ standing for } \frac{e^{-\sigma}-r}{e^{\sigma}-r}\right)$$

where Φ_1 is a function expressible in terms of elliptic integrals as shown in the Note at the end of this paper.

IX. $P_{-\frac{2}{3}}(\cosh\sigma)+\frac{1}{3}\,P_{\frac{1}{3}}(\cosh\sigma).\,r+\frac{1.4}{3.6}\,P_{\frac{4}{3}}(\cosh\sigma).\,r^2$

$+\ldots$to inf. $=\Phi_2$, a function expressible in elliptic integrals.

Proof:

Proceeding as in the case of I, the sum of the series is found to be

$$\frac{1}{\pi}\int_0^{\pi}d\phi\left\{\sum_{m=0}^{\infty}\frac{r^m}{D^{m+\frac{1}{3}}}\cdot\frac{1.4.7\ldots(3m-2)}{3.6.9\ldots3m}\right\}$$

$$=\frac{1}{\pi}\int_0^{\pi}d\phi\Big/(D-r)^{\frac{1}{3}}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\{e^{\sigma}-r-2\sinh\sigma\sin^2\theta\}^{\frac{1}{3}}}$$

$$=\frac{2}{\pi(e^{\sigma}-r)^{\frac{1}{3}}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{3}}}=\Phi_2,$$

$$\left(k^2 \text{ standing for } \frac{2\sinh\sigma}{e^{\sigma}-r}\right)$$

where Φ_2 is well-known to be expressible in terms of elliptic integrals. (See, *e.g.*, Greenhill's book, *l.c.*, p. 165.)

X. $Q_{-\frac{2}{3}}(\cosh\sigma)+\frac{1}{3}.\,Q_{\frac{1}{3}}(\cosh\sigma).\,r+\frac{1.4}{3.6}\,Q_{\frac{4}{3}}(\cosh\sigma).\,r^2$

$+\ldots$to inf. $=\Phi_3$,

where Φ_3 is expessible in terms of elliptic integrals as Φ_1 is.

Proof:

Proceeding as in the case of III, the sum of the series is found to be equal to

$$\int_0^\infty \frac{d\theta}{(\cosh\sigma - r + \sinh\sigma\cosh\theta)^{\frac{1}{3}}}$$

which is seen to be equal to

$$\frac{2}{(e^{\sigma}-r)^{\frac{1}{3}}} \int_0^{\frac{\pi}{2}} \frac{(\cos\phi)^{-\frac{1}{3}}\, d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{3}}}$$

$$\left(k^2 \text{ standing for } \frac{e^{-\sigma}-r}{e^{\sigma}-r} \right)$$

Note. $I \equiv \int_0^{\frac{\pi}{2}} \frac{(\cos\phi)^{-\frac{1}{2}} d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{4}}}$

$$= \sum_{n=0}^{\infty} \left\{ k^{2n} . \frac{1.5.9\cdots(4n-3)}{4.8\ldots 4n} \int_0^{\frac{\pi}{2}} (\cos\phi)^{-\frac{1}{2}} \sin^{2n}\phi\, d\phi \right\}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left\{ k^{2n} . \frac{1.5.9\ldots(4n-3)}{4.8\cdots 4n} \; \frac{\Gamma(\frac{1}{4})\Gamma(\frac{2n+1}{2})}{\Gamma(\frac{2n+1}{2}+\frac{1}{4})} \right\}$$

$$= \frac{\{\Gamma(\frac{1}{4})\}^2}{2\sqrt{2\pi}} . \sum_0^{\infty} \left\{ k^{2n} . \frac{1.5.9\ldots(4n-3)}{3.7\ldots(4n-1} \times \frac{1.3\ldots(2n-1)}{4.8\ldots 4n} . 2^n \right\}$$

$$= \left\{ \Gamma(\tfrac{1}{4})^2 \Big/ 2\sqrt{2\pi} \right\} \times F\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, k^2 \right)$$

But each factor in the above expression is expressible as an elliptic integral. (See Whittaker and Watson's *Modern Analysis*, 4th edition, p. 524, for the first factor, and Kummer's paper on hypergeometric series in *Crelle's Journal*, Bd. 15, p. 146, for the second factor).

Thus it is proved that Φ_1 is expressible in terms of elliptic integrals.

Bull. Cal. Math. Soc., Vol. XXIII, No. 3 (1931).

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BRAHMAGUPTA ON INTERPOLATION

BY

P. C. SENGUPTA, M.A.

(Read, the 14th September, 1930.)

Brahmagupta in his Khaṇḍakhādyaka (665 A. D.) tabulates the "sines" and the equations of apsis of the sun and of the moon at intervals of 15° of arc or of the mean anomaly.

As to the planets Mercury, Venus, Mars, Jupiter and Saturn he tabulates the values of the elongation or of the annual parallax, at irregular intervals of the Sīghra anomaly. This Sīghra anomaly in our way is the exterior angle of elongation at the sun of any planet whether inferior or superior. In the Uttarādhyāya or the concluding chapter of the Khaṇḍakhādyaka, where Brahmagupta, modifies the astronomical constants as taken from Aryabhata's ārdharātrika system of astronomy,* he gives his new rules of interpolation. He gives two rules, one applies to those tables where the argument increases by a common difference of 15°, the other applies to those tables where the argument increases irregularly.

The first one may be thus stated:—

"Multiply the residual arc left after division by 900′ (*i.e.*, 15°), by half the difference of the tabular difference passed over and that to be passed over and divide by 900′ (*i.e.*, 15°): by the result increase or decrease as the case may be, half the sum of the same two tabular differences; the result which, whether less or greater than the tabular difference to be passed over, is the true tabular difference to be passed over." †

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* Aryabhata's Lost work by P. C. Sengupta, *Bull. Cal. Math. Soc.*; also Aryabhata, *Journal of Letters*, Calcutta University, XVIII.

† गतभोग्यखण्डकान्तरदलविकलवधात् शतैर्नवभिराप्ता ।
तद्युतिदलंयुतोनं भोग्यादूनाधिकं भोग्यं ॥

Pandit B. Misra's edition of the Khaṇḍakhādyaka, p. 35.

Illustration :—

Bahmagupta's table of 'sines' (*i.e.*, 150× sine of an arc) in the Khaṇḍakhādyaka is as follows :—

arc	"Sine"	Tabular Difference	Second Difference
0°	0		
15°	39	39	
30°	75	36	—3
45°	106	31	—5
60°	130	24	—7
75°	145	15	—9
90°	150	5	—10

Let it be proposed to find the "sine" of 57°.

Now 57°=3420′=900′×3+720′. Thus three of tabular differences are considered as passed over, the last one being 31; the one to be passed over is 24.

The true tabular difference by the rule, for the arc of 57°

$$= \frac{31+24}{2} - \frac{720}{900} \times \frac{31-24}{2}$$

∴ the "sine" of 57° $=106+\frac{720}{900}\left(\frac{31+24}{2}-\frac{720}{900}\times\frac{31-24}{2}\right)$

$$=125{\cdot}76.$$

As worked out from the logarithm table

$150 \sin 57° = 125{\cdot}80.$

Again "sine" of 57° from Brahmagupta's rule

$$=106+\frac{720}{900}.24+\frac{31-24}{2}\times\frac{720}{900}-\left(\frac{720}{900}\right)^2\times\frac{31-24}{2}$$

$$=106+\frac{720}{900}\times 24+\frac{720}{900}\left(\frac{720}{900}-1\right)\times\frac{24-31}{2}\,*,$$

which is in the modern form of the interpolation equation up to the term containing the second difference.

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* *Cf.* Ball's Spherical Astronomy, p. 18[illegible] etc.

Brahmagupta thus takes a decidedly improved step in the art of interpolation by taking into account the second difference. This illustrates Brahmagupta's first rule.

His second rule may be thus stated :—

"Multiply the increase of the Śīghra anomaly to be passed over by the increase of the Śīghra equation passed over and divide by the increase of the anomaly passed over : the result is the number of degrees in the 'adjusted' increase of the equation passed over. Multiply half the difference of this result and the increase of the Śīghra equation to be passed over by the residue (of the anomaly left after subtracting as many as possible of the preceding intervals of Śīghra anomaly) and divide by the increase of the Śīghra anomaly to be passed over : by the new result decrease or increase as the case may be, half the sum of the same two increases of the equations ; the final result which is either greater or less than the increase of the Śīghra equation to be passed over, is the true increase of the equation to be passed over." *

Illustration :—

Mars' tabulated differences of the Śīghra equation are as follows :—

Increase of the Śīghra anomaly	Increase of the Śīghra Equation
28°	11°
32°	12°
30°	10°
31°	7°
&c.	&c.

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* भुक्तगति फलांश गुणा भोग्यगतिर्भुक्तगति हृतालब्धम्।
भुक्तगतेः फलभोगाद्भोग्य फलान्तरार्धहतम्॥
विकलम् भोग्यगति हृतं लब्धेनोनाधिकं फलैक्यार्धम्।
भोग्यफलादधिकोनं तद्भोग्यफलं स्फुटं भवति॥
Pandit Babua Misra's edition of the Khaṇḍakhādyaka, pp. 66-67.

Here the intervals of anomaly are not equal. Suppose it is proposed to find the Śīghra equation for the anomaly of 74°.

The residue of the Śīghra anomaly

$$=74^\circ-(28^\circ+32^\circ)=14^\circ.$$

The rules given above say that for the last 30° of the Śīghra anomaly the change in the Śīghra equation

$$=\frac{14^\circ\times 30}{32}=13^\circ 7' 30''.$$

Now as in the first rule, the Śīghra equation for 74° of the Śīghra anomaly of Mars

$$=11^\circ+12^\circ+\frac{14}{30}\left\{\frac{13^\circ 7' 30''+10^\circ}{2}-\frac{14}{30}\left(\frac{13^\circ 7' 30''-10^\circ}{2}\right)\right\}$$

$$=28^\circ\ 3'\ 20''.$$

Mars' Śīghra epicycle having a circumference of 234°, the Śīghra equation for 74° of the Śīghra anomaly, accurately worked out becomes

$$=27^\circ\ 55'\ 7''.$$

We have thus seen that Brahmagupta uses the second difference in interpolation, but it appears doubtful if he had any successor to follow him up in this line. Bhāskara II (1150 A.D.) quotes Brahmagupta's rule without any attempt at improvement. Thus in the Grahagaṇita, Spaṣṭādhikāra 16, for the 'sines' which are tabulated at intervals 10°, the same rule is repeated in the following terms :—

"Multiply the residual arc (left after division by 10°) by the difference of the two tabular differences of sines passed over, and to be passed over and divide by twenty; diminish or increase by it the sum of the same two tabular differences for finding the sine or the versed sine of the given arc." *

* यातैष्ययोः खण्डकयोर्विशेषः शेषांशनिघ्नो नखहृत्तदूनम् ।
युतं गतैष्यैक्यदलं स्फुटं स्यात् क्रमोत्क्रमज्याकरणेऽत्र भोग्यम् ॥

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ON A GENERALISED THEOREM ON INTEGRAL INEQUALITY

BY

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(*Read 12th July, 1931.*)

1. The object of the present paper is to give a generalisation of my Fundamental Integral Inequality theorem, which along with several other generalisations and applications appeared in a previous number of this Bulletin.* The relations considered there were between the quotients of the integrals of two or more positive continuous functions and the integrals of the quotients of the functions taken in pairs. The fundamental inequality established there is

$$(1) \quad \int_{x_1}^{x_2} \frac{f(x)}{g(x)}\, dx \leq \text{ or } \geq (x_2 - x_1) \frac{\displaystyle\int_{x_1}^{x_2} f(x)dx}{\displaystyle\int_{x_1}^{x_2} g(x)dx},$$

according as $\dfrac{f(x)}{g(x)}$ and $g(x)$ both increase (or both decrease), or one increases and the other decreases, $f(x)$ and $g(x)$ being any two positive continuous functions defined in the interval $x_1 \leq x \leq x_2$.

In the generalised theorem given below in §1, the quotient under the integral sign is taken of any number of positive continuous functions both in the numerator and in the denominator and the corresponding relation is obtained with the help of the Fundamental Inequality referred to above. In §2, inequalities of Tchebycheff † and others are generalised, and those of Dunkel,[*] Jensen,[†] Hölder,[‡] Schwarz, Fujiwara[§] and Hayashi,[||] are deduced directly from the theorem. In §3, the theorem has been illustrated and applied to analytic, circular and transcendental functions.

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* "On Some Integral Inequalities," *Bulletin of the Calcutta Mathematical Society*, Vol. 22 (1930), pp. 133-152.

† See Hermite, "*Cours professé pendant le 2ᵉ semestre*, 1881-82.

For the sake of brevity, we shall say that two functions are of the same sense, when they both increase or both decrease, and of opposite senses, when one increases and the other decreases, in the given interval.

§1

Theorem.

2. *If* $f_\nu(x)$ *and* $\phi_\nu'(x)$ *be positive continuous functions of* x, *for* $\nu=1, 2, 3, \ldots\ldots n$ *defined in the interval* $x_1 \leq x \leq x_2$, *then will*

$$\int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\ldots f_n(x)}{\phi_1(x)\phi_2(x)\ldots\phi_{n-1}(x)}\,dx$$

$$(1.1) \qquad \leq \text{ or } \geq \frac{\int_{x_1}^{x_2} f_1(x)dx \int_{x_1}^{x_2} f_2(x)dx \ldots\ldots \int_{x_1}^{x_2} f_n(x)dx}{\int_{x_1}^{x_2} \phi_1(x)dx \int_{x_1}^{x_2} \phi_2(x)dx \ldots\ldots \int_{x_1}^{x_2} \phi_{n-1}(x)dx}$$

according as $\dfrac{f_1(x)f\ (x)\ldots f_{\nu-1}(x)}{\phi_1(x)\phi_2(x)\ldots\phi_{\nu-1}(x)}$ and $\dfrac{\phi_{\nu-1}(x)}{f_\nu(x)}$

($\nu=2, 3, \ldots n.$) *are of the same sense or are of opposite senses.*

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[*] O. Dunkel, "Integral inequalities with applications to the calculus of variations," *The American Mathematical Monthly*, Vol. 31 (1924), pp. 326-337.

[†] J. L. W. V. Jensen, "Sur les Fonctions convexes et les inegalites entre les valeurs moyennes," *Acta Math.*, Vol. 30 (1906), pp. 175-193.

[‡] O. Hölder, "Ueber einen Mittelwertsatz," *Nachricaten Ges Wiss. Gottingen*, 1889, pp. 38-47.

[§] M. Fujiwara, ' Ein von Brunn vermuteter Satz uber konvexe Flachen und eine Verallgemeinerung der Schwarzschen und der Tchebycheffschen Ungleichungen fur bestimmte Integrale," *Tohoku Math. Journal*, 13 (1918), pp. 228-235.

And "Über eine Ungleichung fur bestimmte Integrale," *ibid*, 15, (1919), pp. 285-288.

[||] T. Hayashi, "On some inequalities," *Rendiconti del Circolo Matematico di Palermo*, 44 (1920), pp. 336-340, *Proof.*

Consider the integral $\int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\dots f_{n-1}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-1}(x)}\,dx$.

We have by the application of [1],

$$\int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\dots f_{n-1}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-1}(x)}\,d$$

$$(1.2) \qquad \leq \text{ or } \geq (x_2-x_1)\,\frac{\displaystyle\int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\dots f_{n-1}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-2}(x)}\,dx}{\displaystyle\int_{x_1}^{x_2} \phi_{n-1}(x)\,dx}$$

according as $\frac{f_1(x)f_2(x)\dots f_{n-1}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-1}(x)}$ and $\phi_{n-1}(x)$

are of the same sense or of oppposite senses.

Again

$$\int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\dots f_{n-1}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-1}(x)}$$

$$(1.3) \qquad \geq \text{ or } \leq (x_2-x_1)\,\frac{\displaystyle\int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\dots f_n(x)}{\phi_1(x)\phi_2(x)\cdots\phi_{n-1}(x)}\,dx}{\displaystyle\int_{x_1}^{x_2} f_n(x)\,dx}.$$

according as $\frac{f_1(x)f_2(x)\dots f_{n-1}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-1}(x)}$ and $f_n(x)$

are of opposite senses or of the same sense.

Hence combining (1.2) and (1 3) we have

$$\int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\dots f_n(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-1}(x)}\,dx$$

$$(1.4) \qquad \leq \text{ or } \geq \int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\dots f_{n-1}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-2}(x)}\,dx \, \frac{\int_{x_1}^{x_2} f_n(x)dx}{\int_{x_1}^{x_2} \phi_{n-1}(x)dx}$$

according as $\dfrac{f_1(x)f_2(x)\dots f_{n-1}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-1}(x)}$ and $\dfrac{\phi_{n-1}(x)}{f_n(x)}$

are of the same sense or of opposite senses.

From the above it is evident that $f_n(x)$ and $\phi_{n-1}(x)$ are of opposite senses. Next suppose them to be of the same sense.

We have

$$(1.5) \qquad \int_{x_1}^{x_2} \frac{\phi_{n-1}(x)}{f_n(x)}\,dx \geq \text{ or } \leq (x_2-x_1) \frac{\int_{x_1}^{x_2} \phi_{n-1}(x)dx}{\int_{x_1}^{x_2} f_n(x)dx}$$

according as $\dfrac{\phi_{n-1}(x)}{f_n(x)}$ and $f_n(x)$ are of opposite senses or have the same sense.

Again

$$\int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\dots f_n(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-1}(x)}\,dx$$

$$(1.6) \qquad \leq \text{ or } \geq (x_2-x_1) \frac{\int_{x_1}^{x_2} \dfrac{f_1(x)f_2(x)\dots f_{n-1}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-2}(x)}\,dx}{\int_{x_1}^{x_2} \dfrac{\phi_{n-1}(x)}{f_n(x)}\,dx}$$

according as $\dfrac{f_1(x)f_2(x)\dots f_n(x)}{\phi_1(x)\phi_2(x)\cdots\phi_{n-1}(x)}$ and $\dfrac{\phi_{n-1}(x)}{f_n(x)}$

have the same sense, or have opposite senses.

And combining (1.5) and (1.6) we get (1.4) under the same set of conditions, so that (1.4) is true whether $f_n(x)$ and $\phi_{n-1}(x)$ have the same sense or have opposite senses.

In a similar manner, we get

$$\int_{x_1}^{x_2}\frac{f_1(x)f_2(x)\dots f_n(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-1}(x)}\,dx$$

$$\leq \text{ or } \geq \int_{x_1}^{x_2}\frac{f_1(x)f_2(x)\dots f_{n-2}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-3}(x)}\,dx\,\frac{\displaystyle\int_{x_1}^{x_2} f_{n-1}(x)dx\int_{x_1}^{x_2} f_n(x)dx}{\displaystyle\int_{x_1}^{x_2}\phi_{n-2}(x)dx\int_{x_1}^{x_2}\phi_{n-1}(x)dx}$$

where in addition to the conditions for the validity of (1.4), similar conditions hold simultaneously between

$$\frac{f_1(x)f_2(x)\dots f_{n-2}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-2}(x)} \text{ and } \frac{\phi_{n-2}(x)}{f_{n-1}(x)}.$$

Proceeding in this manner, we obtain at the penultimate stage,

$$\int_{x_1}^{x_2}\frac{f_1(x)f_2(x)\dots f_n(x)}{\phi_1(x)\phi_2(x)\cdots\phi_{n-1}(x)}\,dx$$

$$\leq \text{ or } \geq \int_{x_1}^{x_2}\frac{f_1(x)f_2(x)}{\phi_1(x)}dx\,\frac{\displaystyle\int_{x_1}^{x_2} f_3(x)dx\dots\dots\int_{x_1}^{x_2} f_n(x)dx}{\displaystyle\int_{x_1}^{x_2}\phi_2(x)dx\dots\dots\int_{x_1}^{x_2}\phi_{n-1}(x)dx}$$

which again

$$\leq \text{ or } \geq \frac{\int_{x_1}^{x_2} f_1(x)dx \int_{x_1}^{x_2} f_2(x)dx \int_{x_1}^{x_2} f_3(x)dx \ldots\ldots \int_{x_1}^{x_2} f_n(x)dx}{\int_{x_1}^{x_2} \phi_1(x)dx \int_{x_1}^{x_2} \phi_2(x)dx \ldots\ldots \int_{x_1}^{x_2} \phi_{n-1}(x)dx}$$

according as $\dfrac{f_1(x)f_2(x)\cdots f_{\nu-1}(x)}{\phi_1(x)\phi_2(x)\ldots\phi_{\nu-1}(x)}$ and $\dfrac{\phi_{\nu-1}(x)}{f_\nu(x)}$

$$(\nu=2, 3 \ldots\ldots\ldots n)$$

have the same sense or have opposite senses.

Thus our theorem is proved.

Equality occurs either when

$$\frac{f_1(x)}{\phi_1(x)} = \frac{f_2(x)}{\phi_2(x)} = \ldots = \frac{f_{n-1}(x)}{\phi_{n-1}(x)},$$

or when all the f's and all the ϕ's are constants.

3. *Note* 1.—Putting $n+m$ for n and taking $\phi_n(x)$, $\phi_{n+1}(x), \ldots$ $\phi_{n+m-1}(\;)$ to be constants, we have

$$\int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\ldots f_n(x)\cdots f_{n+m}(x)}{\phi_1(x)\phi_2(x)\ldots\phi_{n-1}(x)}\,dx$$

$$(1.7) \quad \leq \text{ or } \geq \frac{1}{(x_2-x_1)^m} \frac{\int_{x_1}^{x_2} f_1(x)dx \ldots \int_{x_1}^{x_2} f_n(x)dx \ldots \int_{x_1}^{x_2} f_{n+m}(x)dx}{\int_{x_1}^{x_2} \phi_1(x)dx \ldots\ldots \int_{x_1}^{x_2} \phi_{n-1}(x)dx}$$

Similarly

$$\int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\ldots f_{n-1}(x)}{\phi_1(x)\cdots\phi_n(x)\ldots\phi_{n+m}(x)}\,dx$$

$$(1.8) \quad \leq \text{or} \geq (x_2-x_1)^{m+2} \frac{\int_{x_1}^{x_2} f_1(x)dx \ldots\ldots \int_{x_1}^{x_2} f_{n-1}(x)dx}{\int_{x_1}^{x_2} \phi_1(x)dx \ldots \int_{x_1}^{x_2} \phi_n(x)dx \ldots\ldots \int_{x_1}^{x_2} \phi_{n+m}(x)dx}$$

Note 2.—If $f_n(x)$ become constant in (1.1), we have

$$\int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\dots f_{n-1}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{n-1}(x)}\,dx$$

$$\text{(1.9)} \qquad \leq \text{ or } \geq (x_2-x_1)\,\frac{\int_{x_1}^{x_2} f_1(x)dx\dots\int_{x_1}^{x_2} f_{n-1}(x)dx}{\int_{x_1}^{x_2} \phi_1(x)dx\dots\int_{x_1}^{x_2} \phi_{n-1}(x)dx}$$

Note 3.—If m ϕ's are constant in (1.8) (as no generality is lost, we take $\phi_{n+1}, \phi_{n+2},\dots\phi_{n+m}$ to be constant), we have

$$\int_{x_1}^{x_2} \frac{f_1(x)f_2(x)\dots\dots f_{n-1}(x)}{\phi_1(x)\phi_2(x)\dots\dots\phi_n(x)}\,dx$$

$$\text{(1.10)} \qquad \leq \text{ or } \geq (x_2-x_1)^2\,\frac{\int_{x_1}^{x_2} f_1(x)dx\dots\dots\int_{x_1}^{x_2} f_{n-1}(x)dx}{\int_{x_1}^{x_2} \phi_1(x)dx\dots\dots\int_{x_1}^{x_2} \phi_n(x)\,dx}$$

§ 2

Derivations and Generalisations of known Inequalities.

4. In this section I propose to generalise certain well-known inequalities, namely, those of Tchebycheff and others and deduce those of Dunkel, Jensen, Hölder, Schwarz, Fujiwara and Hayashi directly from my theorem.

5. *Generalisation of Tchebycheff's inequality.*

Taking all the ϕ's to be constant in (1.1), we have

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$$\int_{x_1}^{x_2} f_1(x)\,f_2(x)\dots f_n(x)\,dx$$

$$\text{(2.1)} \qquad > \frac{1}{(x_2-x_1)^{n-1}}\int_{x_1}^{x_2} f_1(x)\,dx\int_{x_1}^{x_2} f_2\,(x)\,dx\dots\int_{x_1}^{x_2} f_n(x)\,dx$$

Here all the f's are of the same sense.

In particular, when all the f's are equal, we have

$$(2.2)\quad \int_{x_1}^{x_2} \{f(x)\}^n dx > \frac{1}{(x_2-x_1)^{n-1}} \left\{ \int_{x_1}^{x_2} f(x)\, dx \right\}^n .$$

These are generalised forms of Tchebycheff's inequality.*

6. *Generalisation of a previous result and derivation of Dunkel's theorem.*

If all the f's are constant in (1. 1), we obtain

$$\int_{x_1}^{x_2} \frac{dx}{\phi_1(x)\, \phi_2(x) \ldots \phi_{n-1}(x)}$$

$$(2.3)\qquad > \frac{(x_2-x_1)^n}{\int_{x_1}^{x_2} \phi_1(x)\, dx \int_{x_1}^{x_2} \phi_2(x)\, dx \ldots \int_{x_1}^{x_2} \phi_{n-1}(x)\, dx}$$

where all the ϕ's have the same sense.†

In particular if $\phi_1 = \phi_2 = \ldots = \phi_{n-1}$,

we have Dunkel's theorem

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* See O. Dunkel's paper, l. c.

† An exception to this condition that all the ϕ's should have the same sense, can be noticed as follows :—

Take $n=3$, and let $\phi_1(x)$ and $\phi_2(x)$ have opposite senses in the interval.

$$\text{Then} \int_{x_1}^{x_2} \frac{dx}{\phi_1(x)\phi_2(x)} > (x_2-x_1) \frac{\int_{x_1}^{x_2} \frac{dx}{\phi_2(x)}}{\int_{x_1}^{x_2} \phi_1(x)\, dx},$$

when $\phi_1(x)$ and $\phi_1(x)\phi_2(x)$ have the same sense, and this is greater than

$$\frac{(x_2-x_1)^3}{\int_{x_1}^{x_2} \phi_1(x)\, dx \int_{x_1}^{x_2} \phi_2(x)\, dx}.$$

A similar reasoning will hold when $\phi_2(x)$ and $\phi_1(x)\phi_2(x)$ have the same sense.

$$(2.4)\qquad \int_{x_1}^{x_2} \frac{dx}{\{\phi(x)\}^{n-1}} \left\{\int_{x_1}^{x_2} \phi(x)\, dx\right\}^{n-1} > (x_2-x_1)^n.$$

The results (2. 3) and (2. 4) can be taken as generalisations of the result (1. 6) of my previous paper.*

Again combining (2. 1) with (2. 3), and (2. 2) with (2. 4) we obtain

$$(x_2-x_1)^{n-1}\int_{x_1}^{x_2} f_1(x)\, f_2(x) \ldots f_n(x)\, dx$$

$$(2.5)\qquad > \int_{x_1}^{x_2} f_1(x)\, dx \int_{x_1}^{x_2} f_2(x)\, dx \ldots \int_{x_1}^{x_2} f_n(x)\, dx$$

$$> \frac{(x_2-x_1)^{n+1}}{\displaystyle\int_{x_1}^{x_2} \frac{dx}{f_1(x)\, f_2(x) \ldots f_n(x)}}$$

and $$(x_2-x_1)^{n-1}\int_{x_1}^{x_2} \{f(x)\}^n dx > \left\{\int_{x_1}^{x_2} f(x)\, dx\right\}^n \ldots\ldots,$$

$$(2.6)\qquad > \frac{(x_2-x_1)^{n+1}}{\displaystyle\int_{x_1}^{x_2} \frac{dx}{\{f(x)\}^n}}.$$

7. *Derivation of Jensen's Inequality.*

If we put $\qquad f_1 = f_2 = \ldots = f_n = fg$

and $\qquad \phi_1 = \phi_2 = \ldots = \phi_{n-1} = g,$

in the result (1.1), where f and g are positive, we have

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* *Loc. cit.*, p. 137. See also (2.8) at p. 148, where (2.4) has been deduced from another theorem of mine. See also Dunkel's paper, *loc. cit.*, wherein (2.4) appears as a theorem.

$$\frac{f_1 f_2 \ldots f_n}{\phi_1 \phi_2 \ldots \phi_{n-1}} = g f^n,$$

and $$\frac{f_1 f_2 \ldots f_{\nu-1}}{\phi_1 \phi_2 \ldots \phi_{\nu-1}} = f^{\nu-1} \text{ and } \frac{\phi_{\nu-1}}{f_\nu} = \frac{1}{f}, \qquad (\nu=2, 3, \ldots, n)$$

whence by the application of (1.1), we have

$$\int_{x_1}^{x_2} g(x)\{f(x)\}^n dx > \frac{\left\{ \int_{x_1}^{x_2} f(x) g(x) dx \right\}^n}{\left\{ \int_{x_1}^{x_2} g(x) dx \right\}^{n-1}}$$

i.e.,

$$(2.7) \qquad \left\{ \int_{x_1}^{x_2} f(x) g(x) dx \right\}^n < \int_{x_1}^{x_2} g(x)\{f(x)\}^n dx \left\{ \int_{x_1}^{x_2} g(x) dx \right\}^{n-1}$$

which is the inequality due to Jensen.*

8. *Derivation of Hölder's and Schwarz's Inequalities.*

Again if we put

$$f_1 = f_2 = f_3 = \ldots = f_n = fg$$

and $\phi_1 = \phi_2 = \phi_3 = \ldots = \phi_{n-1} = g^{\frac{n}{n-1}}$, in the result (1.1),

then $$\frac{f_1 f_2 \ldots f_n}{\phi_1 \phi_2 \ldots \phi_{n-1}} = f^n,$$

and $$\frac{f_1 f_2 \ldots f_{\nu-1}}{\phi_1 \phi_2 \ldots \phi_{\nu-1}} = \left\{ \frac{f}{g^{\frac{1}{n-1}}} \right\}^{\nu-1} \text{ and } \frac{\phi_{\nu-1}}{f_\nu} = \frac{g^{\frac{1}{n-1}}}{f}$$

$(\nu=2, 3, \ldots n)$

---

* J. L. W. V. Jensen, *l.c.*, p. 187.

Therefore

$$\int_{x_1}^{x_2}\{f(x)\}^n dx > \frac{\left\{\int_{x_1}^{x_2} f(x)g(x)dx\right\}^n}{\left[\int_{x_1}^{x_2}\{g(x)\}^{\frac{n}{n-1}}dx\right]^{n-1}}$$

or

$$(2.8)\qquad \int_{x_1}^{x_2} f(x)g(x)dx$$

$$< \left[\int_{x_1}^{x_2}\{f(x)\}^n dx\right]^{\frac{1}{n}}\left[\int_{x_1}^{x_2}\{g(x)\}^{\frac{n}{n-1}}dx\right]^{\frac{n-1}{n}},$$

which is Hölder's inequality.*

Putting $n=2$, we get Schwarz's inequality

$$(2.9)\qquad \int_{x_1}^{x_2} f(x)g(x)dx < \left[\int_{x_1}^{x_2}\{f(x)\}^2 dx\right]^{\frac{1}{2}}\left[\int_{x_1}^{x_2}\{g(x)\}^2 dx\right]^{\frac{1}{2}}.$$

9. *Derivation of Fujiwara's Inequality.*

Put $n=3$ and take f_3 to be a constant, then we have

$$(2.10)\qquad \int_{x_1}^{x_2}\frac{f_1(x)f_2(x)}{\phi_1(x)\phi_2(x)}\,dx$$

$$\leq \text{ or } \geq (x_2-x_1)\,\frac{\int_{x_1}^{x_2} f_1(x)dx\int_{x_1}^{x_2} f_2(x)dx}{\int_{x_1}^{x_2}\phi_1(x)dx\int_{x_1}^{x_2}\phi_2(x)dx},$$

according as $\frac{f_1(x)f_2(x)}{\phi_1(x)\phi_2(x)}$ and $\phi_2(x)$ are of the same sense or of opposite senses, and $\frac{f_1(x)}{\phi_1(x)}$ and $\frac{\phi_1(x)}{f_2(x)}$ are of the same sense or of opposite senses.

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* O. Hölder, *l.c.*

Now put $f_1=F_1\Phi_1, f_2=F_2\Phi_2,$

$\phi_1=F_1\Phi_2, \phi_2=F_2\Phi_1,$ in (2.10),

whence

$$\int_{x_1}^{x_2} dx \le \text{or} \ge (x_2-x_1)\frac{\int_{x_1}^{x_2}F_1(x)\Phi_1(x)dx\int_{x_1}^{x_2}F_2(x)\Phi_2(x)dx}{\int_{x_1}^{x_2}F_1(x)\Phi_2(x)dx\int_{x_1}^{x_2}F_2(x)\Phi_1(x)dx}$$

i.e., $\int_{x_1}^{x_2}F_1(x)\Phi_2(x)dx\int_{x_1}^{x_2}F_2(x)\Phi_1(x)$

$$(2.11) \qquad \le \text{or} \ge \int_{x_1}^{x_2}F_1(x)\Phi_1(x)dx\int_{x_1}^{x_2}F_2(x)\Phi_2(x)dx,$$

according as $\frac{F_1(x)}{F_2(x)}$ and $\frac{\Phi_1(x)}{\Phi_2(x)}$ have the same sense, or have opposite senses.

This inequality is due to Fujiwara.*

10. *Derivation of Hayashi's Inequality.*†

Again, Hayashi's inequality

$$\int_{x_1}^{x_2} p_1(x)p_2(x)\dots p_n(x)dx\left\{\int_{x_1}^{x_2} g_1(x)g_2(x)\dots g_n(x)dx\right\}^{n-1}$$

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* M. Fujiwara, *l.c.*

See also my paper, *l.c.*, pp. 149-150, wherein it has been deduced in another way from the Fundamental Theorem (1).

† T. Hayashi, *l.c.*

$$(2.12) \quad \geq \int_{x_1}^{x_2} p_1(x)g_2(x)\dots g_n(x)dx \int_{x_1}^{x_2} g_1(x)p_2(x)g_3(x)\dots g_n(x)dx\dots\dots$$

$$\dots\dots \int_{x_1}^{x_2} g_1(x)g_2(x)\cdots g_{n-1}(x)p_n(x)dx$$

is at once deduced from (1.1) by putting
$f_1(x)=p_1(x)p_2(x)\dots p_n(x),\ f_2(x)=f_3(x)=\dots=f_n(x)=g_1(x)g_2(x)\dots g_n(x)$
$\phi_1(x)=p_1(x)g_2(x)\dots g_n(x),\ \phi_2(x)=g_1(x)p_2(x)g_3(x)\dots g_n(x)$
$\phi_{n-1}(x)=g_1(x)g_2(x)\dots p_{n-1}(x)g_n(x).$
for then

$$\frac{f_1(x)f_2(x)\dots f_{\nu-1}(x)}{\phi_1(x)\phi_2(x)\dots\phi_{\nu-1}(x)} = \frac{p_\nu(x)p_{\nu+1}(x)\dots p_n(x)}{g_\nu(x)g_{\nu+1}(x)\dots g_n(x)}$$

and

$$\frac{\phi_{\nu-1}(x)}{f_\nu(x)} = \frac{p_{\nu-1}(x)}{g_{\nu-1}(x)}$$

so that the result (2.12) holds when $\dfrac{p_\nu(x)}{g_\nu(x)}$ have the same sense for $\nu=1, 2, 3\dots n$.

§ 3

Illustrations and Applications.

11. *Illustrations.*

(1) Suppose $f_1(x)=x^m$, $f_2(x)=x^n$ and $\phi_1(x)=x^p$, where m, n, p are positive numbers. Then by (1.1)

$$\int_0^x x^{m+n-p}dx < \text{or} > \frac{\int_0^x x^m dx \int_0^x x^n dx}{\int_0^x x^p dx},$$

according as $m-p$ and $p-n$ have the same sign or opposite signs.

Thus

$$\int_0^x x^2 dx < \frac{\int_0^x x^3 dx \int_0^x x\, dx}{\int_0^x x^2 dx}.$$

(2). Similarly,

$$\int \tan^2 x\,dx > \frac{\int \tan x\,dx \int \sec x\,dx}{\int \operatorname{cosec} x\,dx}$$

the integrations extending over any suitable interval within 0 and $\frac{\pi}{2}$.

In particular,

$$(3.1) \qquad \frac{2}{\sqrt{3}} - \frac{\pi}{6} > \log \sqrt{3}.$$

12. Consider the integral $\int \frac{dx}{(x^2+1)(x^2-x+1)}$

Here the function x^2-x+1 changes sense at $x=\frac{1}{2}$, but x^2+1 retains the same sense in the interval $0 \leq x \leq 1$.

Taking the interval $(\frac{1}{2}, 1)$, we have

$$\int_{\frac{1}{2}}^{1} \frac{dx}{(x^2+1)(x^2-x+1)} > \frac{\frac{1}{8}}{\int_{\frac{1}{2}}^{1} (x^2+1)dx \int_{\frac{1}{2}}^{1} (x^2-x+1)dx}$$

whence

$$(3.2) \qquad \log \frac{\sqrt{6}}{\sqrt{5}} + \frac{1}{\sqrt{3}} \frac{\pi}{6} > \frac{36}{95}$$

Similarly

$$\int_0^{\frac{1}{2}} \frac{dx}{(x^2+1)(x^2-x+1)} > \frac{\frac{1}{8}}{\int_0^{\frac{1}{2}} (x^2+1)dx \int_0^{\frac{1}{2}} (x^2-x+1)dx}$$

so that

$$(3.3) \quad \frac{1}{2} \log \frac{5}{3} - \frac{1}{\sqrt{3}} \frac{5\pi}{6} > \frac{36}{65}.$$

Again

$$\log\sqrt{2}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}$$

$$-\frac{1}{\sqrt{3}}\tan^{-1}\frac{-1}{\sqrt{3}}>\frac{36\times 32}{5\times 13\times 19}.$$

$$\text{(3.4)}\quad\text{or}\quad \log\sqrt{2}-\frac{2\pi}{3\sqrt{3}}>\frac{36\times 32}{5\times 13\times 19}.$$

13. For the integral $\int_0^1 \frac{dx}{(1+x^2)(1-x^2)^{\frac{1}{2}}}$,

we have

$$\int_0^1 \frac{dx}{(1+x^2)(1-x^2)^{\frac{1}{2}}}>\frac{1}{\int_0^1 (1+x^2)\,dx\int_0^1 \sqrt{1-x^2}\,dx}$$

whence we get

$$\text{(3.5)}\qquad \frac{\pi}{2\sqrt{2}}>\frac{3}{\pi},$$

$$\frac{\pi^2}{4}>\frac{3}{\sqrt{2}}$$

14. We have

$$\int_0^{\frac{\pi}{4}} d\theta>\frac{\left(\frac{\pi}{4}\right)^3}{\int_0^{\frac{\pi}{4}}\sqrt{\tan\theta}\;d\theta\int_0^{\frac{\pi}{4}}\sqrt{\cot\theta}\;d\theta},$$

whence follows

$$\text{(3.6)}\qquad\left[\frac{\pi}{2}-\log(\sqrt{2}+1)\right]\left[\frac{\pi}{2}-\log(\sqrt{2}-1)\right]>\frac{\pi^2}{8}.$$

15. Next consider the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-nx}\cos^n x\,dx$$

where m and n are positive integers of which m is even.

We easily get

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-nx} \cos^m x \, dx = \int_0^{\frac{\pi}{2}} e^{-nx} \cos^m x \, dx$$

$$+\int_0^{\frac{\pi}{2}} e^{nx} \cos^m x \, dx$$

which, by the application of the theorem

$$> \frac{\int_0^{\frac{\pi}{2}} \cos^{\frac{m}{2}} x \, dx \int_0^{\frac{\pi}{2}} \cos^{\frac{m}{2}} x \, dx}{\int_0^{\frac{\pi}{2}} e^{nx} dx} + \frac{\int_0^{\frac{\pi}{2}} \cos^{\frac{m}{2}} x \, dx \int_0^{\frac{\pi}{2}} \cos^{\frac{m}{2}} x \, dx}{\int_0^{\frac{\pi}{2}} e^{-nx} dx}.$$

Hence we get

$$\frac{m(m-1)(m-2)(m-3)\ldots\ldots 2.1}{(n^2+m^2)\{n^2+(m-2)^2\}\ldots\ldots(n^2+2^2)n^2}$$

$$(3.7) \qquad > \left\{ \frac{(m-2)(m-6)(m-10)\ldots 3.1}{m(m-4)(m-8)\ldots 4.2} \, \frac{\pi}{2} \right\}^2 \frac{1}{2\cosh\frac{n\pi}{2}-2}$$

when $\frac{m}{2}$ is even ;

$$(3.8) \quad \text{and} \qquad > \left\{ \frac{(m-2)(m-6)(m-10)\ldots 4.2}{m(m-4)(m-8)\ldots 5.3} \right\}^2 \frac{1}{2\cosh\frac{n\pi}{2}-2}$$

when $\frac{m}{2}$ is odd.

16. Consider the integral, $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2-b^2\cos^2 x}$

where $a>b$, a and b being positive numbers.

We have

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a^2-b^2\cos^2 x} > \left(\frac{\pi}{2}\right)^3 \frac{1}{\int_0^{\frac{\pi}{2}} (a-b\cos x)dx \int_0^{\frac{\pi}{2}} (a+b\cos x)dx}$$

$$i.e., \quad > \frac{\pi^3}{2} \frac{1}{(a\pi-2b)\,(a\pi+2b)}\,.$$

Hence we get

$$\frac{1}{a\sqrt{a^2-b^2}}\,\frac{\pi}{2} > \frac{\pi^3}{2}\cdot\frac{1}{(a\pi-2b)\,(a\pi+2b)}$$

$$(3.9) \quad \text{or} \quad \frac{1}{a\sqrt{a^2-b^2}} > \frac{\pi^2}{(a\pi-2b)(a\pi+2b)}\,.$$

In particular, we get,

$$\frac{1}{3.5} > \frac{\pi^2}{(5\pi-8)(5\pi+8)}\,;$$

$$(3.10) \quad \frac{1}{4.5} > \frac{\pi^2}{(5\pi-6)(5\pi+6)}\,;$$

$$\frac{1}{2\sqrt{3}} > \frac{\pi^2}{(2\pi-2)(2\pi+2)}\,;$$

$$\frac{2}{\sqrt{3}} > \frac{\pi^2}{\pi^2-1}\,.$$

17. *Inequalities in Series.*

Take the case of the integral* $\int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{(a+\sin x)(b+\sin x)}$

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[*] This integral was considered in my previous paper, l.c., para. pp. 140-141, with the help of the fundamental theorem, the results wherein can well be compared with the results given here.

where $b > a \geq 1$.

With the help of our theorem, it can be easily seen that

$$\int_0^{\frac{\pi}{2}} \frac{\cos x\,dx}{(a+\sin x)(b+\sin x)} > \left(\frac{\pi}{2}\right)^2 \cdot \frac{\int_0^{\frac{\pi}{2}} \cos x\,dx}{\int_0^{\frac{\pi}{2}} (a+\sin x)dx \int_0^{\frac{\pi}{2}} (b+\sin x)dx}$$

$$> \left(\frac{\pi}{2}\right)^2 \cdot \frac{1}{\left(a\frac{\pi}{2}+1\right)\left(b\frac{\pi}{2}+1\right)}$$

i.e. $$> \frac{\pi^2}{(a\pi+2)(b\pi+2)}.$$

Hence we have

$$(3.11)\quad \frac{1}{a(b+1)} - \frac{1}{2}\,\frac{(b-a)}{a^2(b+1)^2} + \frac{1}{3}\,\frac{(b-a)^2}{a^3(b+1)^3} - \ldots\ldots > \frac{\pi^2}{(a\pi+2)(b\pi+2)}$$

In particular, we have

$$\frac{1}{3} - \frac{1}{2.3^2} + \frac{1}{3 \cdot 3^3} - \frac{1}{4.3^4} + \ldots\ldots > \frac{\pi^2}{2(\pi+1)(\pi+2)}\ ;$$

$$\frac{1}{5} - \frac{1}{2.5^2} + \frac{1}{3.5^3} - \frac{1}{4.5^4} + \ldots\ldots > \frac{\pi^2}{2(\pi+1)(2\pi+1)}\ ;$$

$$(3.12)\quad \frac{3}{5} - \frac{1.3^2}{2.5^2} + \frac{1.3^3}{3.5^3} - \frac{1.3^4}{4.5^4} + \ldots\ldots > \frac{3\pi^2}{2(\pi+2)(2\pi+1)}\ ;$$

$$\frac{4}{2.7} - \frac{1}{2} \cdot \frac{4^2}{2^2.7^2} + \frac{1}{3} \cdot \frac{4^3}{2^3.7^3} - \frac{1}{4} \cdot \frac{4^4}{2^4.7^4} + \ldots > \frac{\pi^2}{(\pi+1)(3\pi+1)}.$$

Many such interesting results can be obtained easily.

18. Again from the integral $\int_a^b \sec^{2n+2}\theta\,d\theta$,

where $0 \leq a < b \leq \frac{\pi}{2}$ we get the following interesting results.

$$(3.13)\quad 1+{}^nC_1\,\frac{1}{3}+{}^nC_2\,\frac{1}{5}+\ldots+\frac{1}{2n+1} > 2^{n+1}\left(\frac{\pi}{4}\right)^{2n+3}.$$

$$1+{}^nC_1\,\frac{3}{3}+{}^nC_2\cdot\frac{3^2}{5}+\ldots+\frac{3^n}{2n+1} > 2^{2n+2}\left(\frac{\pi}{3\sqrt{3}}\right)^{2n+3};$$

$$\frac{1}{\sqrt{3}}+{}^nC_1\,\frac{1}{3.3\sqrt{3}}+{}^nC_2\,\frac{1}{5.3^2\sqrt{3}}+\ldots+\frac{1}{(2n+1,3^n\sqrt{3}}$$

$$> \frac{1}{2}\cdot\left(\frac{\pi}{3}\right)^{2n+2}.$$

Similarly from the integral $\int_a^b \frac{dx}{\sec^{2n+1}x}$,

we get

$$(3.14)\ \frac{\sqrt{3}}{2}-{}^nC_1\,\frac{3\sqrt{3}}{3.2^3}+{}^nC_2\,\frac{3^2.\sqrt{3}}{5.2^5}-\ldots\ldots+(-1)^n\frac{3^n\sqrt{3}}{(2n+1).2^{2n+1}}$$

$$> \left(\frac{\pi}{3}\right)^{2n+2}\frac{1}{\{\log(\sqrt{3}+1)-\log(\sqrt{3}-1\}^{2n+1}}$$

$$\frac{1}{2}-{}^nC_1\,\frac{1}{3.2^3}+{}^nC_2\,\frac{1}{5.2^5}-\ldots+(-1)^n\frac{1}{(2n+1).2^{2n+1}}$$

$$> \left(\frac{\pi}{6}\right)^{2n+2}\frac{1}{(\log\sqrt{3})^{2n+1}}.$$

and from the integral $\int_a^b \frac{dx}{\cot^{2n}x}$, we have

$$(3.15)\ \frac{1}{2n-1}\,\frac{3^{2n-1}-1}{3^{n-1}\sqrt{3}}-\frac{1}{2n-3}\,\frac{3^{2n-3}-1}{3^{n-2}\sqrt{3}}+\frac{1}{2n-5}\cdot\frac{3^{2n-5}-1}{3^{n-3}\sqrt{3}}-\ldots\ldots$$

$$+(-1)^{n-1}\frac{3-1}{\sqrt{3}}+(-1)^n\frac{\pi}{6} > \left(\frac{\pi}{6}\right)^{2n+1}\frac{1}{(\log\sqrt{3})^{2n}};$$

$$\frac{1}{2n-1}\,\frac{3^{n-1}\sqrt{3}-1}{3^{n-1}\sqrt{3}}-\frac{1}{2n-3}\,\frac{3^{n-2}\sqrt{3}-1}{3^{n-2}\sqrt{3}}+\ldots\ldots+(-1)^{n-1}\frac{\sqrt{3}-1}{\sqrt{3}}$$

$$+(-1)^n\frac{\pi}{12} > \left(\frac{\pi}{12}\right)^{2n+1}\frac{1}{(\log\sqrt{2})^{2n}};$$

$$\frac{1}{2n-1}(3^{n-1}\sqrt{3}-1)-\frac{1}{2n-3}(3^{n-2}\sqrt{3}-1)+\frac{1}{2n-5}(3^{n-3}\sqrt{3}-1)-\ldots$$

$$+(-1)^{n-1}(\sqrt{3}-1)+(-1)^n\frac{\pi}{12} > \left(\frac{\pi}{12}\right)^{2n+1}\frac{1}{(\log\sqrt{3}-\log\sqrt{2})^{2n}}.$$

19. *Applications to Circular Functions of Special Type.*

Here we shall consider integrals of the form *

$$\int\frac{f(\sin\theta,\cos\theta)\,d\theta}{\cos(\theta-\alpha_1)\cos(\theta-\alpha_2)\ldots\ldots\cos(\theta-\alpha_n)},$$

where $f(\sin\theta,\cos\theta)$ is any homogeneous function of $\sin\theta$, $\cos\theta$, of $n-1$ dimensions, and $0 < \alpha_\nu < \frac{\pi}{2}$ for $\nu=1, 2, \ldots n$.

By the application of the theorem, we have

$$(3.16)\qquad \int_0^{\frac{\pi}{2}}\frac{f(\sin\theta,\cos\theta)\,d\theta}{\cos(\theta-\alpha_1)\cos(\theta-\alpha_2)\ldots\cos(\theta-\alpha_n)}$$

$$< \text{ or } > \left(\frac{\pi}{2}\right)^n\frac{\int_0^{\frac{\pi}{2}}f(\sin\theta,\cos\theta)\,d\theta}{(\cos\alpha_1+\sin\alpha_1)(\cos\alpha_2+\sin\alpha_2)\ .\ (\cos\alpha_n+\sin\alpha_n)}$$

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* Analogous integrals with

$$\prod_1^n\sin(\theta-\alpha_\nu),\ \text{ or }\ \prod_1^n\sin(\theta+\alpha_\nu)\ \text{ or }\ \prod_1^n\cos(\theta+\alpha_\nu)$$

in the denominator can be treated similarly provided the range of integration a values of α_ν are suitably chosen.

according as $\dfrac{f(\sin\theta, \cos\theta)}{\prod\limits_{\nu=1}^{\nu} \cos(\theta-\alpha_\nu)}$ and $\cos(\theta-\alpha_\nu)$ $(\nu=1, 2, 3, \ldots\ldots n)$

are of the same sense or of opposite senses.

Hence, we obtain

$$(3.17)\quad \sum_{\nu=1}^{n} \frac{f(-\cos\alpha_\nu, \sin\alpha_\nu)}{\sin(\alpha_\nu-\alpha_1)\sin(\alpha_\nu-\alpha_2)\ldots{}^{*}\ldots\sin(\alpha_\nu-\alpha_n)} \times$$

$$\log\frac{\sin\alpha_\nu+\cos\alpha_\nu+1}{\sin\alpha_\nu+\cos\alpha_\nu-1}$$

$$< \text{or} > \left(\frac{\pi}{2}\right)^n \frac{\int_0^{\frac{\pi}{2}} f(\sin\theta, \cos\theta)\, d\theta}{(\cos\alpha_1+\sin\alpha_1)(\cos\alpha_2+\sin\alpha_2)\ldots(\cos\alpha_n+\sin\alpha_n)},$$

the sign * denoting that the factor $\sin(\alpha_\nu-\alpha_\nu)$ has been omitted from the denominator of the above co-efficient.

When $f(\sin\theta, \cos\theta)$ can be broken up into factors the result can also be modified in the light of notes 2 and 3.

As a particular case consider the integral

$$\int_0^{\frac{\pi}{2}} \frac{\sin^3\theta\, d\theta}{\cos\left(\theta-\frac{5\pi}{12}\right)\cos\left(\theta-\frac{\pi}{3}\right)\cos\left(\theta-\frac{\pi}{6}\right)\cos\left(\theta-\frac{\pi}{12}\right)}$$

which is

$$> \left(\frac{\pi}{2}\right)^4 \frac{\int_0^{\frac{\pi}{2}} \sin^3\theta\, d\theta}{\left(\cos\frac{5\pi}{12}+\sin\frac{5\pi}{12}\right)\left(\cos\frac{\pi}{3}+\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{6}+\sin\frac{\pi}{6}\right)} \times$$

$$\frac{1}{\left(\cos\frac{\pi}{12}+\sin\frac{\pi}{12}\right)} \quad \text{by } (3\cdot 16)$$

$$\text{(3.18)} \qquad \text{i.e.,} > \frac{\pi^4}{9(\sqrt{3}+1)^2}$$

$$> \frac{\pi^4}{18(2+\sqrt{3})} \quad ; \text{ i.e.,} > \pi^4\left(\frac{1}{9} - \frac{1}{6\sqrt{3}}\right).$$

It can also be shown that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^3\theta \, d\theta}{\cos\left(\theta-\frac{5\pi}{12}\right)\cos\left(\theta-\frac{\pi}{3}\right)\cos\left(\theta-\frac{\pi}{6}\right)\cos\left(\theta-\frac{\pi}{12}\right)}$$

$$\text{(3.19)} \qquad > \frac{\pi^2}{3(2+\sqrt{3})}.$$

20. *An Inequality Involving Successive Differentiations.*

From the integral $\int_a^x \frac{dx}{(x+c)^{p+1}(x-a)^{\frac{2n+1}{2}}}$

where c and a are positive numbers, and p and n are positive integers, we get after some easy calculations

$$\frac{2^{n+1}(-1)^p}{p!\ 1.3.5\ldots(2n-1)} \frac{\partial^{n+p}}{\partial c^p \partial a^n} \left\{ \frac{1}{\sqrt{c+a}} \tan^{-1} \sqrt{\frac{x-a}{c+a}} \right\}$$

$$\text{(3.20)} \qquad > \frac{(p+2)(2n+3)}{2\{(x+c)^{p+2}-(a+c)^{p+2}\}(x-a)^{\frac{2n-3}{2}}}.$$

21. *Applications to Elliptic Functions.*

Consider the integral $\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-k^2\sin^2\phi)^3}$

where k is the modulus of the elliptic function.

It can be easily seen that

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-k^2\sin^2\phi)^3} > \left(\frac{\pi}{2}\right)^7 \frac{1}{E^6}$$

whence we have

$$(3.21) \qquad \frac{8-8k^2+3k^4}{8k'^5} > \left(\frac{\pi}{2}\right)^6 \frac{1}{E^6}$$

Again from the inequality

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-k^2\sin^2\phi)^3} \left\{\int_0^{\frac{\pi}{2}} (1-k^2\sin^2\phi)d\phi\right\}^3 > \left(\frac{\pi}{2}\right)^4$$

we get

$$(3.22) \qquad \frac{8-8k^2+3k^4}{8k'^5} \left(1-\frac{k^2}{2}\right)^3 > 1.$$

22. Again from the integral $\int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$

we have

$$(3.23) \quad \mathrm{sn}^{-1}z > \frac{4kz^3}{(z\sqrt{1-z^2}+\sin^{-1}z)(kz\sqrt{1-k^2z^2}+\sin^{-1}kz)}$$

In particular taking $z=1$, we have

$$K > \frac{8k}{\pi(kk'+\sin^{-1}k)},$$

$$(3.24) \quad i.e., \frac{\pi}{2}K > \frac{4k}{kk'+\sin^{-1}k}.$$

Similarly

$$\frac{\pi}{2}K' > \frac{4k'}{kk'+\sin^{-1}k'}.$$

whence

$$(3.25) \qquad \left(\frac{\pi}{2}\right)^2 KK' > \frac{16kk'}{(kk'+\sin^{-1}k)(kk'+\sin^{-1}k')},$$

also we get

$$(3.26) \qquad \left(\frac{\pi}{2}\right)^2 \left(EK'+EK'-\frac{\pi}{2} \right)$$

$$> \frac{16kk'}{(kk'+\sin^{-1}k)(kk'+\sin^{-1}k')}$$

$$(3.27) \qquad \text{or } (EK'+E'K-K'K)^2 KK' > \frac{16kk'}{(kk'+\sin^{-1}k)(kk'+\sin^{-1}k')}.$$

23. Next consider the integral $\int_0^K \mathrm{nd}^2 u\, du$.

We easily get

$$\int_0^K \mathrm{nd}^2 u\, du > \frac{K^2}{\int_0^K (1+k \operatorname{sn} u) du \int_0^K (1-k \operatorname{sn} u) du}$$

whence

$$(3.28) \qquad \frac{E}{k'^2} > \frac{K^2}{\left(K+\log \frac{1+k'+k}{1+k'-k} \right)\left(K-\log \frac{1+k'+k}{1+k'-k} \right)}$$

Again

$$\int_0^K \mathrm{dn}^2 u\, du < \frac{1}{K}\left(K+\log \frac{1+k'+k}{1+k'-k} \right)\left(K-\log \frac{1+k'+k}{1+k'-k} \right)$$

or

$$(3.29) \qquad EK < \left(K+\log \frac{1+k'+k}{1+k'-k} \right)\left(K-\log \frac{1+k'+k}{1+k'-k} \right)$$

Hence combining (3.28) and (3.29), we have

$$\frac{K^4k'^2}{\left(K+\log\frac{1+k'+k}{1+k'-k}\right)\left(K-\log\frac{1+k'+k}{1+k'-k}\right)} < EK$$

$$(3.30)\qquad < \left(K+\log\frac{1+k'+k}{1+k'-k}\right)\left(K-\log\frac{1+k'+k}{1+k'-k}\right);$$

or

$$(3.31)\qquad \frac{K^2k'^2}{\left(K+\log\frac{1+k'+k}{1+k'-k}\right)\left(K-\log\frac{1+k'+k}{1+k'-k}\right)} < \frac{E}{K}$$

$$< \frac{\left(K+\log\frac{1+k'+k}{1+k'-k}\right)\left(K-\log\frac{1+k'+k}{1+k'-k}\right)}{K^2}.$$

whence

$$(3.32)\qquad \left(K+\log\frac{1+k'+k}{1+k'-k}\right)\left(K-\log\frac{1+k'+k}{1+k'-k}\right) > K^2k',$$

i.e.,
$$\left\{\frac{\pi}{2}\,\vartheta_3^2(0,\tau)+\log\frac{\vartheta_2^2(0\,\tau)+\vartheta_3^2(0|\tau)+\vartheta_4^2(0|\tau)}{\vartheta_3^2(0|\tau)+\vartheta_4^2(0|\tau)-\vartheta_2^2(0|\tau)}\right\}$$

$$(3.33)\qquad \times\left\{\frac{\pi}{2}\,\vartheta_3^2(0|\tau)-\log\frac{\vartheta_2^2(0|\tau)+\vartheta_3^2(0|\tau)+\vartheta_4^2(0|\tau)}{\vartheta_3^2(0|\tau)+\vartheta_4^2(0|\tau)-\vartheta_2^2(0|\tau)}\right\}$$

$$> \frac{\pi^2}{4}\,\vartheta_3^2(0|\tau)\vartheta_4^2(0|\tau).$$

Again combining the inequality *

$$(3.34)\qquad 1-\frac{k^2}{2} > \frac{E}{K}$$

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* See the last line of p. 148 of my previous paper.

with (3.31), we have

$$\left\{\frac{\pi}{2}\,\vartheta_3^2(0|\tau)+\log\frac{\vartheta_2^2(0|\tau)+\vartheta_3^2(0|\tau)+\vartheta_4^2(0|\tau)}{\vartheta_3^2(0|\tau)+\vartheta_4^2(0|\tau)-\vartheta_2^2(0|\tau)}\right\}$$

$$(3.35)\qquad\times\left\{\frac{\pi}{2}\,\vartheta_3^2(0|\tau)-\log\frac{\vartheta_2^2(0|\tau)+\vartheta_3^2(0|\tau)+\vartheta_4^2(0|\tau)}{\vartheta_3^2(0|\tau)+\vartheta_4^2(0|\tau)-\vartheta_2^2(0|\tau)}\right\}$$

$$>\frac{\pi^2}{2}\,\frac{\vartheta_3^4(0|\tau)\vartheta_4^4(0|\tau)}{\vartheta_3^4(0|\tau)+\vartheta_4^4(0|\tau)}$$

Also we have

$$(3.36)\qquad\frac{\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-k^2u)^{\frac{1}{2}}du}{\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-k^2u)^{-\frac{1}{2}}du}$$

$$<\frac{\left(K+\log\frac{1+k'+k}{1+k'-k}\right)\left(K-\log\frac{1+k'+k}{1+k'-k}\right)}{K^2}$$

and

$$\pi^2<\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-k^2u)^{-\frac{1}{2}}du\int_0^1 u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-k^2u)^{\frac{1}{2}}du$$

$$(3.37)\qquad<4\left(K+\log\frac{1+k'+k}{1+k'-k}\right)\left(K-\log\frac{1+k'+k}{1+k'-k}\right).$$

In conclusion, I wish to express my best thanks to Professor Ganesh Prasad for his kind interest and constant encouragement.

ON THE SUMMATION OF INFINITE SERIES OF LEGENDRE'S FUNCTIONS.

BY

N. G. SHABDE.

(*Calcutta University.*)

(*Read, the 23rd August, 1931.*)

Introduction.—The object of the present paper is to sum such infinite series of Legendre's functions, P_n $(\cosh \sigma)$ or Q_n $(\cosh \sigma)$, with non-integral n, as admit of being summed up into forms, compact and *free from the sign of integration*

The first successful attempt at the summation of such series has been made by Professor Ganesh Prasad,* who has recently given ten such sums.

Many of the series given below are believed to be new and they are starred. There are also a number of other series, which deserve special attention, because, although they can be obtained as deductions from the results of Professor Prasad or by methods quite analogous to those used by him, their sums are so simple and interesting that they merit to be added to the list. These series are not starred.

My sincere thanks are due to Professor Ganesh Prasad for suggesting to me this continuation of his work and for his kind and keen interest in it throughout its course.

Taking $K_p\ (\cosh \psi) = P_{-\frac{1}{2}+p}\ (\cosh \psi)$,

$$(1)\quad K_1\ (\cosh \psi) + \frac{1}{3}\ K_3\ (\cosh \psi) - \frac{1}{5}\ K_5\ (\cosh \psi) - \frac{1}{7}\ K_7\ (\cosh \psi) + + - - + + \cdot \text{ to inf.}$$

$$= \frac{1}{\sqrt{2}}\ k^{\frac{1}{2}} F(k'),\ k = e^{\psi}, \text{ for } 0 < \psi < \frac{\pi}{4},$$

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* "On the summation of infinite series of Legendre's functions," (second paper), *Bulletin of the Calcutta Mathematical Society*, Vol. XXIII, No. 3, pp. 115-124.

and

$$=\frac{1}{\sqrt{2}}\,k^{\frac{1}{2}}\,F(k'),\quad k=e^{\frac{\pi}{4}}\quad \text{for } \frac{\pi}{4}<\psi<\frac{3\pi}{4}.$$

Proof:

It is known * that

$$K_p\,(\cosh\psi)=\frac{2}{\pi}\int_0^{\psi}\frac{\cos p\phi\cdot d\phi}{\{2\,(\cosh\psi-\cosh\phi)\}^{\frac{1}{2}}}.$$

Therefore, the series in question has the sum

$$\frac{2}{\pi}\int_0^{\psi}\frac{d\phi}{\{2\,(\cosh\psi-\cosh\phi)\}^{\frac{1}{2}}}\left\{\cos\phi+\frac{1}{3}\cos 3\phi-\frac{\cos 5\phi}{5}\right.$$

$$\left.-\frac{\cos 7\phi}{7}++\ldots\ldots \text{ to inf.}\right\}. \quad (i)$$

The cosine † series within the crooked brackets equals $\frac{\pi}{4}\sqrt{2}$ when

$$-\frac{\pi}{4}<\phi<\frac{\pi}{4} \text{ and equals zero when } \frac{\pi}{4}<\phi<\frac{3\pi}{4}.$$

Therefore, (*i*) equals

$$\frac{2}{\pi}\cdot\frac{\pi}{4}\cdot\sqrt{2}\cdot\int_0^{\psi}\frac{d\phi}{\{2\,(\cosh\psi-\cosh\phi)\}^{\frac{1}{2}}}$$

$=\frac{1}{\sqrt{2}}\,k^{\frac{1}{2}}\,F\,(k')$, $k=e^{\psi}$, assuming ψ to lie between zero and $\frac{\pi}{4}$

$$\text{and } =\frac{1}{\sqrt{2}}\,k^{\frac{1}{2}}\,F\,(k'),\quad k=e^{\frac{\pi}{4}},\quad \frac{\pi}{4}<\psi<\frac{3\pi}{4}.$$

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* See Hobson's paper, "On a type of spherical harmonics of unrestricted degree, order, and argument," *Phil. Trans. of the Royal Society of London, A.*, Vol. 187, 1896, p. 580.

† *Encyklop. der Math. Wissensch.* II—I, p. 938.

$$(2)\quad K_1(\cosh\psi)-\frac{1}{5}K_5(\cosh\psi)+\frac{1}{7}K_7(\cosh\psi)-\frac{K_{11}(\cosh\psi)}{11}+-\ldots\text{to inf.}$$

$$=\frac{1}{\sqrt{3}}k^{\frac{1}{3}}F(k'),\quad k=e^{\psi},\text{ for }0<\psi<\frac{\pi}{3}$$

and

$$=\frac{1}{\sqrt{3}}k^{\frac{1}{3}}F(k'),\quad k=e^{\frac{\pi}{3}},\quad\text{for }\frac{\pi}{3}<\psi<\frac{2\pi}{3}.$$

Proof:

Proceeding as in (1) we find that the series has the sum

$$\frac{2}{\pi}\int_0^{\psi}\frac{d\phi}{\{2(\cosh\psi-\cosh\phi)\}^{\frac{1}{2}}}\left\{\cos\phi-\frac{\cos 5\phi}{5}+\frac{\cos 7\phi}{7}-\frac{1}{11}\cos 11\phi+-+\ldots\text{to inf.}\right\}\qquad\text{(i)}$$

The series * within the crooked brackets has the sum $\frac{1}{\sqrt{3}}\cdot\frac{3}{2}\cdot\frac{\pi}{3}$ when $0<\phi<\frac{\pi}{3}$ and zero when $\frac{\pi}{3}<\phi<\frac{2\pi}{3}$.

Therefore (i) equals $\frac{1}{\sqrt{3}}\int_0^{\psi}\frac{d\phi}{\{2(\cosh\psi-\cosh\phi)\}^{\frac{1}{2}}}$

$$=\frac{1}{\sqrt{3}}k^{\frac{1}{3}}F(k'),\quad k=e^{\psi}\text{ and }0<\psi<\frac{\pi}{3}.$$

When $\frac{\pi}{3}<\psi<\frac{2\pi}{3}$, (i) equals $\frac{1}{\sqrt{3}}k^{\frac{1}{3}}F(k')$, $k=e^{\frac{\pi}{3}}$.

$$(3)\quad P_{-\frac{1}{3}}(\cosh\sigma)+\frac{1}{9}P_{\frac{1}{3}}(\cosh\sigma)+\frac{1\cdot 4}{9\cdot 18}P_{\frac{1}{3}}(\cosh\sigma)+\ldots\quad\text{to inf.}$$

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* Whittaker and Watson, *Modern Analysis*, Third Edition, p. 192, Ex. 16.

$$=\frac{2}{6^{\frac{2}{3}}}\cdot\frac{1}{\pi}\cdot\frac{\Gamma^2(\frac{1}{6})}{\Gamma(\frac{1}{3})}\cdot\frac{1}{(3\sqrt{17}-5)^{\frac{1}{3}}}, \text{ where } \cosh\sigma=\frac{5\cdot\sqrt{17}-3}{16}.$$

Proof:

$$^{*}\mathrm{P}_{-\frac{1}{3}}(\cosh\sigma)+\frac{1}{3}\,r\,\mathrm{P}_{\frac{1}{3}}(\cosh\sigma)+\frac{1\cdot4}{3\cdot6}\,r^2\,\mathrm{P}_{\frac{4}{3}}(\cosh\sigma)+\ldots$$

$$\text{to inf., } r<e^{-\sigma}$$

$$=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{(\cosh\sigma-r+\sinh\sigma\cdot\cos\phi)^{\frac{1}{3}}}$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1}{(e^{\sigma}-r)^{\frac{1}{3}}}\cdot\frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{3}}},\ k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}\ \text{(Putting } \phi=2\theta)$$

$$=\frac{2}{\pi(e^{\sigma}-r)^{\frac{1}{3}}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{3}}}=\frac{2}{\pi(e^{\sigma}-r)^{\frac{1}{3}}}\int_0^{\mathrm{K}}\mathrm{dn}^{\frac{1}{3}}u\,du.$$

Put $r=\frac{1}{3}$, $k^2=\frac{3}{4}$ and hence $e^{\sigma}=\frac{-1+\sqrt{17}}{2}$ and $\cosh\sigma=\frac{5\sqrt{17}-3}{16}$

and we have

$$\mathrm{P}_{-\frac{1}{3}}(\cosh\sigma)+\frac{1}{9}\,\mathrm{P}_{\frac{1}{3}}(\cosh\sigma)+\frac{1\cdot4}{9\cdot18}\,\mathrm{P}_{\frac{4}{3}}(\cosh\sigma)+\ldots\ \text{to inf.}$$

$$=\frac{2}{\pi}\,\frac{6^{\frac{1}{3}}}{(3\sqrt{17}-5)^{\frac{1}{3}}}\cdot\int_0^{\mathrm{K}}\sqrt[3]{\mathrm{dn}\,u}\,du,\ k=\frac{\sqrt{3}}{2}$$

$$=\frac{2}{\pi}\cdot\frac{6^{\frac{1}{3}}}{6\cdot(3\sqrt{17}-5)^{\frac{1}{3}}}\,\frac{\Gamma^2(\frac{1}{6})}{\Gamma(\frac{1}{3})}.$$

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[*] Ganesh Prasad, *l. c.*, p. 128, Series IX.

(4) $Q_{-\frac{1}{2}}(2)+\frac{1}{2+\sqrt{3}}\cdot\frac{1}{2}Q_{\frac{1}{2}}(2)+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{(2+\sqrt{3})^{2}}Q_{\frac{3}{2}}(2)+\ldots$ to inf.

$$=\frac{\pi}{(2\sqrt{3})^{\frac{1}{2}}}$$

Proof :

* $Q_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}Q_{\frac{1}{2}}(\cosh\sigma)\,r+\ldots$ to inf.

$$=\frac{2K}{\sqrt{(e^{\sigma}-r)}}\text{, modulus of K being }\sqrt{\left(\frac{e^{-\sigma}-r}{e^{\sigma}-r}\right)},\ r<e^{\sigma}.$$

Putting $\cosh\sigma=2$ and $\sqrt{\left(\frac{e^{-\sigma}-r}{e^{\sigma}-r}\right)}=0$,

we have $e^{\sigma}=2+\sqrt{3}$, $r=\frac{1}{e^{\sigma}}=\frac{1}{2+\sqrt{3}}$ and $K=\frac{\pi}{2}$.

Therefore,

$$Q_{-\frac{1}{2}}(2)+\frac{1}{2+\sqrt{3}}\,\frac{1}{2}\cdot Q_{\frac{1}{2}}(2)+\ldots\quad\text{to inf.}$$

$$=\frac{\pi}{(2\sqrt{3})^{\frac{1}{2}}}.$$

(5) $Q_{-\frac{1}{2}}(\cosh 2)+\frac{1}{e^{2}}\cdot\frac{1}{2}\cdot Q_{\frac{1}{2}}(\cosh 2)+\frac{1.3}{2.4}\,\frac{1}{e^{4}}Q_{\frac{3}{2}}(\cosh 2)+\ldots$ to inf.

$$=\frac{\pi}{\sqrt{2\sinh 2}}$$

Proof :

Proceeding as in (4) and putting

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* Ganesh Prasad, *l.c.*, p. 117, Series III.

$\sigma=2$ and $\sqrt{\left(\frac{e^{-\sigma}-r}{e^{\sigma}-r}\right)}=0$, we have the required sum.

(6) $Q_{-\frac{1}{2}}(\cosh 1)+\frac{1}{e}\cdot\frac{1}{2}\cdot Q_{\frac{1}{2}}(\cosh 1)+\frac{1.3}{2.4}\,\frac{1}{e^2}\,Q_{\frac{3}{2}}(\cosh 1)$

$+\ldots$to inf.

$$=\frac{\pi}{(2\sinh 1)^{\frac{1}{2}}}$$

Proof :

Proceeding as in (4) and putting

$\sigma=1$ and $\sqrt{\left(\frac{e^{-\sigma}-r}{e^{\sigma}-r}\right)}=0$, we have the required sum.

(7) $(\log 2).Q_{-\frac{1}{2}}(\cosh\sigma)-Q_{\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,Q_{\frac{3}{2}}(\cosh\sigma)-\frac{1}{3}\,Q_{\frac{5}{2}}(\cosh\sigma)$

$+\ldots$to inf.

$$=\frac{1}{\cosh\frac{\sigma}{2}}\left\{-\frac{1}{2}\,\mathrm{K}\log\frac{1}{k}+\frac{\pi}{4}\mathrm{K}'\right\},\ k=\operatorname{sech}\frac{\sigma}{2}.$$

Proof:

It is known[*] that

$$Q_{m-\frac{1}{2}}(\cosh\sigma)=\int_0^\pi\frac{(-1)^m\cos m\theta.d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}.$$

Therefore the sum of the series is equal to

$$\int_0^\pi\frac{d\theta}{\sqrt{2}\,(\cosh\sigma+\cos\theta)}\left\{\log 2+\sum_1^\infty\frac{\cos m\theta}{m}\right\}$$

$$=-\frac{1}{2}\int_0^\pi\frac{\log\sin\frac{\theta}{2}\,.\,d\theta}{\sqrt{\cosh^2\frac{\sigma}{2}-\sin^2\frac{\theta}{2}}},\ \text{as}\ \log 2+\sum_1^\infty\frac{\cos m\theta}{m}=-\log\sin\frac{\theta}{2}$$

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[*] Hobson, *l. c.*, p. 524.

$$=-\int_0^{\frac{\pi}{2}} \frac{\log \sin \phi \, d\phi}{\sqrt{\cosh^2\frac{\sigma}{2}-\sin^2\phi}} \quad \text{putting } \theta=2\phi)$$

$$=-\frac{1}{\cosh\frac{\sigma}{2}}\int_0^{\frac{\pi}{2}} \frac{\log \sin \phi . d\phi}{\sqrt{1-k^2 \sin^2\phi}} \quad \left(\text{where } k^2=\operatorname{sech}^2\frac{\sigma}{2}\right)$$

$$=-\frac{1}{\cosh\frac{\sigma}{2}}\left\{\frac{1}{2}\,\mathrm{K}\log\frac{1}{k}-\frac{\pi}{4}\mathrm{K}'\right\}.$$

(8) $(\log 2)\,Q_{-\frac{1}{2}}(\cosh\sigma)+Q_{\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,Q_{\frac{3}{2}}(\cosh\sigma)$

$+\frac{1}{3}\,Q_{\frac{5}{2}}(\cosh\sigma)+\ldots\ldots$to inf.

$$=-\frac{1}{\cosh\frac{\sigma}{2}}\left\{\frac{\mathrm{K}}{2}\log\frac{k'}{k}-\frac{\pi}{4}\mathrm{K}'\right\}.$$

Proof:

Proceeding as in (7), the sum of the series is found to be equal to

$$-\frac{1}{\cosh\frac{\sigma}{2}}\int_0^{\frac{\pi}{2}} \frac{\log\cos\phi}{\sqrt{1-k^2\sin^2\phi}}\,d\phi, \quad k^2=\operatorname{sech}^2\frac{\sigma}{2}$$

$$=-\frac{1}{\cosh\frac{\sigma}{2}}\left[\frac{\mathrm{K}}{2}\log\frac{k'}{k}-\frac{1}{4}\pi\mathrm{K}'\right]$$

(9) $Q_{\frac{1}{2}}(\cosh\sigma)+\frac{1}{3}\,Q_{\frac{5}{2}}(\cosh\sigma)+\frac{1}{5}\,Q_{\frac{9}{2}}(\cosh\sigma)+\ldots$ to inf.

$$=\frac{1}{4\cosh\frac{\sigma}{2}}\,\mathrm{K}\log\frac{1}{k'}, \quad k=\operatorname{sech}\frac{\sigma}{2}$$

Proof :

Subtract (7) from (8) and the result follows immediately.

*(10) $$rP_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,\frac{r^2}{2}P_{\frac{1}{2}}(\cosh\sigma)+\frac{1.3}{2.4}\cdot\frac{r^3}{3}\,P_{\frac{3}{2}}(\cosh\sigma)$$

$$+\ldots\text{to inf., } r<e^{-\sigma}.$$

$$=\frac{4}{\pi}\left[\sqrt{e^{\sigma}}\,E\left(\sqrt{\frac{2\sinh\sigma}{e^{\sigma}}}\right)\right.$$

$$\left.-\sqrt{(e^{\sigma}-r)}\,E\left(\sqrt{\frac{2\sinh\sigma}{e^{\sigma}-r}}\right)\right].$$

Proof :

It is known† that

$$P_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,P_{\frac{1}{2}}(\cosh\sigma)\,.\,r+\frac{1.3}{2.4}\,r^2P_{\frac{3}{2}}(\cosh\sigma)$$

$$+\ldots\text{to inf., } r<e^{-\sigma}$$

$$=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\sqrt{\cosh\sigma-r+\sinh\sigma\cos\phi}}.$$

Integrating this series with respect to r from $r=0$ to $r=r$ we get

$$r.P_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,\frac{r^2}{2}\,P_{\frac{1}{2}}(\cosh\sigma)+\frac{1.3}{2.4}\,\frac{r^3}{3}\,P_{\frac{3}{2}}(\cosh\sigma)$$

$$+\ldots\ldots\text{to inf.}$$

$$=\frac{1}{\pi}\int_0^{\pi}d\phi\int_0^{r}\frac{dr}{\sqrt{D-r}}=-\frac{2}{\pi}\int_0^{\pi}d\phi\left[\sqrt{D-r}\,\right]_0^r$$

$$=\frac{2}{\pi}\int_0^{\pi}d\phi\,.\,\sqrt{D}-\frac{2}{\pi}\int_0^{\pi}d\phi.\sqrt{D-r}$$

(D standing for $\cosh\sigma+\sinh\sigma\cos\phi$)

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† Ganesh Prasad, *l.c.*, p. 115.

$$=\frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{e^{\sigma}}\, d\theta.\ \sqrt{\left(1-\frac{2\sinh\sigma}{e^{\sigma}}\sin^2\theta\right)}$$

$$=\frac{4}{\pi}\int_0^{\frac{\pi}{2}} d\theta.\sqrt{e^{\sigma}-r}\ \sqrt{1-\left(\frac{2\sinh\sigma}{e^{\sigma}-r}\right)\sin^2\theta}\ \text{(Putting } \phi=2\theta)$$

$$=\frac{4}{\pi}\left[\sqrt{e^{\sigma}}\mathrm{E}\left(\sqrt{\frac{2\sinh\sigma}{e^{\sigma}}}\right)-\sqrt{e^{\sigma}-r}\,\mathrm{E}\left(\sqrt{\frac{2\sinh\sigma}{e^{\sigma}-r}}\right)\right].$$

*(11) $Q_{-\frac{1}{2}}(\cosh\sigma)+rQ_{\frac{1}{2}}(\cosh\sigma)+r^2Q_{\frac{3}{2}}(\cosh\sigma)+\ldots$to inf., $|r|<1$

$$=\frac{k}{2}\left[\mathrm{K}+\frac{1-r^2}{(1+r)^2}\,\Pi\right], n=-\frac{4r}{(1+r)^2}, \text{and } k=\operatorname{sech}\frac{\sigma}{2}.$$

Proof:

It is known † that

$$Q_{m-\frac{1}{2}}(\cosh\sigma)=\int_0^{\pi}\frac{(-1)^m\cos m\theta\, d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}.$$

It is also known that

$$\frac{1+r\cos\theta}{1+2r\cos\theta+r^2}=1-r\cos\theta+r^2\cos 2\theta-+\ldots\text{to inf., } |r|<1.$$

Therefore, the sum of the series in question is easily found to be equal to

$$\int_0^{\pi}\frac{d\theta\cdot(1+r\cos\theta)}{\{1+2r\cos\theta+r^2\}\sqrt{2(\cosh\sigma+\cos\theta)}}$$

$$=\frac{1}{2}\int_0^{\pi}\frac{d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}+\frac{1}{2}(1-r^2)$$

$$\times\int_0^{\pi}\frac{1}{1+2r\cos\theta+r^2}\cdot\frac{d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}$$

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† Hobson, l.c., p. 524.

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{d\phi}{(\cosh^2\frac{\sigma}{2}-\sin^2\phi)^{\frac{1}{2}}}+\frac{1}{2}\frac{(1-r^2)}{(1+r)^2}\int_0^{\frac{\pi}{2}}\frac{d\phi}{(1+n\sin^2\phi)\sqrt{\cosh^2\frac{\sigma}{2}-\sin^2\phi}}$$

(Putting $\theta=2\phi$)

$$=\frac{k}{2}\left[K+\frac{(1-r^2)}{(1+r)^2}\Pi\right], \text{ where } n=-\frac{4r}{(1+r)^2} \text{ and } k=\operatorname{sech}\frac{\sigma}{2}.$$

*(12) $Q_{-\frac{1}{2}}(\cosh\sigma)-2rQ_{\frac{1}{2}}(\cosh\sigma)+2r^2Q_{\frac{3}{2}}(\cosh\sigma)-+\ldots$ to inf. $|r|<1$

$$=\frac{1-r^2}{(1-r)^2}k\Pi,\quad n=\frac{4r}{(1-r)^2} \text{ and } k=\operatorname{sech}\frac{\sigma}{2}.$$

Proof :

$$Q_{m-\frac{1}{2}}(\cosh\sigma)=\int_0^{\pi}\frac{(-1)^m\cos m\theta\,d\theta}{\sqrt{2(\cosh\sigma+\cos\theta)}}.$$

Also

$$\frac{1-r^2}{1-2r\cos\theta+r^2}=1+2r\cos\theta+2r^2\cos 2\theta+2r^3\cos 3\theta+\ldots \text{to inf. } |r|<1.$$

Therefore, the required sum is equal to

$$\int_0^{\pi}\frac{(1-r^2)d\theta}{(1-2r\cos\theta+r^2)\sqrt{2(\cosh\sigma+\cos\theta)}}$$

$$=\int_0^{\frac{\pi}{2}}\frac{(1-r^2)}{(1-r)^2}\frac{d\phi}{\left(1+\frac{4r}{(1-r)^2}\sin^2\phi\right)\sqrt{\cosh^2\frac{\sigma}{2}-\sin^2\phi}}$$

(Putting $\theta=2\phi$)

$$=\frac{1-r^2}{(1-r)^2}k\Pi$$

*(13) $P_{-\frac{1}{2}}(\cosh\sigma)+r\,P_{\frac{1}{2}}(\cosh\sigma)+r^2\,P_{\frac{3}{2}}(\cosh\sigma)+\ldots$ to inf., $r<e^{-\sigma}$

$$=\frac{2}{\pi\sqrt{e^{\sigma}}}\left[K+\frac{r}{e^{\sigma}-r}\Pi\right],$$

the modulus corresponding to K and Π, the complete elliptic integrals, and the parameter n corresponding to Π being respectively $\sqrt{\frac{2\sinh\sigma}{e^{\sigma}}}$

and $\frac{-2\sinh\sigma}{e^{\sigma}-r}$.

Proof :

It is known † that

$$P_{m-\frac{1}{2}}(\cosh\sigma)=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\{\cosh\sigma+\sinh\sigma\cos\phi\}^{m+\frac{1}{2}}}.$$

Therefore the series has the sum

$$\frac{1}{\pi}\int_0^{\pi}d\phi\sum_{m=0}^{\infty}\left\{\frac{r^m}{D^{m+\frac{1}{2}}}\right\},\ (\text{D standing for }\cosh\sigma+\sinh\sigma\cos\phi)$$

$$=\frac{1}{\pi}\int_0^{\pi}d\phi\frac{1}{D^{\frac{1}{2}}}\left\{\frac{D}{D-r}\right\}=\frac{1}{\pi}\int_0^{\pi}d\phi\frac{1}{D^{\frac{1}{2}}}+\frac{r}{\pi}\int_0^{\pi}\frac{d\phi}{D^{\frac{1}{2}}(D-r)}$$

$$=\frac{2}{\pi\sqrt{e\sigma}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}+\frac{2r}{\pi(e^{\sigma}-r)\sqrt{e\sigma}}$$

$$\times\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}},\quad\left(k^2=\frac{2\sinh\sigma}{e^{\sigma}}\text{ and }n=\frac{-2\sinh\sigma}{e^{\sigma}-r}\right)$$

(Putting $2\theta=\phi$)

$$=\frac{2}{\pi\sqrt{e\sigma}}\left[K+\frac{r}{e^{\sigma}-r}\Pi\right]$$

*(14) $P_{\frac{1}{2}}(\cosh\sigma)+2rP_{\frac{3}{2}}(\cosh\sigma)+3r^2P_{\frac{5}{2}}(\cosh\sigma)+\ldots$ to inf., $r<e^{-\sigma}$

$$=\frac{2}{\pi\sqrt{e\sigma}(e^{\sigma}-r)}\left[\Pi+\frac{r.n}{(e^{\sigma}-r)\ 2(1+n)\left(1+\frac{k^2}{n}\right)}\left\{\frac{E}{n}-\frac{k^2+n}{n^2}K\right.\right.$$

$$\left.\left.+\left(1+\frac{2(1+k^2)}{n}+\frac{3k^2}{n^2}\right)\Pi\right\}\right],$$

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† Hobson, *l.c.*, p. 524.

n and k having the same meaning as in (10).

Proof :

Differentiating (13) with respect to r, we have the required sum

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(e^{\sigma}-r)\sqrt{e^{\sigma}}\{1+n\sin^2\theta\}\{1-k^2\sin^2\theta\}^{\frac{1}{2}}}$$

$$+\frac{2r}{\pi\sqrt{e^{\sigma}}(e^{\sigma}-r)^2}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1+n\sin^2\theta)^2\{1-k^2\sin^2\theta\}^{\frac{1}{2}}}$$

$$=\frac{2}{\pi\sqrt{e^{\sigma}}(e^{\sigma}-r)}\left[\Pi+\frac{r}{(e^{\sigma}-r)}\cdot\frac{n}{2(1+n)\left(1+\frac{k^2}{n}\right)}\left\{\frac{E}{n}-\frac{k^2+n}{n^2}K\right.\right.$$

$$\left.\left.+\left(1+\frac{2(1+k^2)}{n}+\frac{3k^2}{n^2}\right)\Pi\right\}\right]$$

(For the second integral see Cayley's *Elliptic Functions*, p. 133.)

$$*(15)\quad P_{\frac{1}{2}}(\cosh\sigma)+\frac{3}{2}rP_{\frac{3}{2}}(\cosh\sigma)+\frac{3.5}{2.4}r^2P_{\frac{5}{2}}(\cosh\sigma)+\ldots$$

to inf., $r<e^{-\sigma}$

$$=\frac{2}{\pi}\frac{E}{(e^{\sigma}-r)^{\frac{3}{2}}k'^2}\text{ where }k'^2=1-k^2=1-\frac{2\sinh\sigma}{e^{\sigma}-r}.$$

Proof :

$$P_{n-\frac{1}{2}}=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\{\cosh\sigma+\sinh\sigma\cos\phi\}^{n+\frac{1}{2}}}.$$

Therefore the series has the sum

$$\frac{1}{\pi}\int_0^{\pi}d\phi\left\{\frac{1}{D^{\frac{1}{2}}}+\frac{3}{2}\cdot\frac{r}{D^{\frac{3}{2}}}+\frac{3.5}{2.4}\frac{r^2}{D^{\frac{5}{2}}}+\ldots\text{to inf.}\right\}$$

(D standing for $\cosh\sigma+\sinh\sigma\cos\phi$)

$$=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{(\mathrm{D}-r)^{\frac{3}{2}}}$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\{e^{\sigma}-r-2\sinh\sigma\sin^2\theta\}^{\frac{3}{2}}}\quad\left(\text{Putting } \theta=\frac{\phi}{2}\right)$$

$$=\frac{2}{\pi(e^{\sigma}-r)^{\frac{3}{2}}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}},\quad\left(k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}\right)$$

$$=\frac{2}{\pi(e^{\sigma}-r)^{\frac{3}{2}}}\int_0^{\mathrm{K}}\frac{du}{\mathrm{dn}^2 u}$$

$$=\frac{2}{\pi(e^{\sigma}-r)^{\frac{3}{2}}}\int_0^{\mathrm{K}}\frac{\mathrm{dn}^2 u\,du}{k'^2}$$ (See *Modern Analysis*, Third Edition, p. 516.)

$$=\frac{2}{\pi(e^{\sigma}-r)^{\frac{3}{2}}}\;\frac{1}{k'^2}\;\mathrm{E},\quad k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}.$$

*(16) $\mathrm{Q}_{\frac{1}{2}}(\cosh\sigma)+\frac{3}{2}\,r\mathrm{Q}_{\frac{3}{2}}(\cosh\sigma+\frac{3.5}{2.4}\,r^2\mathrm{Q}_{\frac{5}{2}}(\cosh\sigma)+\ldots$ to inf., $r<e^{\sigma}$

$$=\frac{2}{(e^{\sigma}-r)^{\frac{3}{2}}}\left[\frac{\mathrm{K}-\mathrm{E}}{k^2}\right],\ k^2=\frac{e^{-\sigma}-r}{e^{\sigma}-r}.$$

Proof.

It is known † that

$$\mathrm{Q}_{n-\frac{1}{2}}(\cosh\sigma)=\int_0^{\infty}\frac{d\theta}{(\cosh\sigma+\sinh\sigma\cosh\theta)^{n+\frac{1}{2}}}$$

Therefore proceeding as in (15), we have the sum of the series equal to

$$\int_0^{\infty}\frac{d\theta}{(\cosh\sigma-r+\sinh\sigma\cosh\theta)^{\frac{3}{2}}}$$

$$=2\int_0^{\infty}\frac{dt}{(\cosh\sigma-r+\sinh\sigma\cosh 2t)^{\frac{3}{2}}},\ \text{where } \theta=2t$$

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† See Hobson, *l.c.*, p. 524.

$$=2\int_0^\infty \frac{dt}{(2\sinh\sigma\cosh^2 t-r+e^{-\sigma})^{\frac{3}{2}}}$$

$$=2\int_0^{\frac{\pi}{2}} \frac{\sec\phi\, d\phi}{(2\sinh\sigma\sec^2\phi-r+e^{-\sigma})^{\frac{3}{2}}}\ (\text{where } \cosh t=\sec\phi)$$

$$=2\int_0^{\frac{\pi}{2}} \frac{\sec\phi\, d\phi}{(\sec\phi)^3\,\{e^{\sigma}-r-(e^{-\sigma}-r)\sin^2\phi\}^{\frac{3}{2}}}$$

$$=\frac{2}{(e^{\sigma}-r)^{\frac{3}{2}}}\int_0^{\frac{\pi}{2}} \frac{\cos^2\phi\, d\phi}{\{1-k^2\sin^2\phi\}^{\frac{3}{2}}},\ k^2=\frac{e^{-\sigma}-r}{e^{\sigma}-r}$$

$$=\frac{2}{(e^{\sigma}-r)^{\frac{3}{2}}}\int_0^{K} \operatorname{cd}^2 u.du$$

$$=\frac{2}{(e^{\sigma}-r)^{\frac{3}{2}}}\left[\left\{\frac{u+k^2\operatorname{sn}u.\operatorname{cd}u}{k^2}\right\}_0^K-\frac{1}{k^2}\int_0^K \operatorname{dn}^2 u du\right]^{\dagger}$$

$$=\frac{2}{(e^{\sigma}-r)^{\frac{3}{2}}}\left[\frac{K-E}{k^2}\right],\ k^2=\frac{e^{-\sigma}-r}{e^{\sigma}-r}.$$

*(17) $$P_{\frac{3}{2}}(\cosh\sigma)+\frac{5}{2}\,r\,P_{\frac{5}{2}}(\cosh\sigma)+\frac{5\cdot7}{2\cdot4}\,r^2\,P_{\frac{7}{2}}(\cosh\sigma)$$

$$+\ldots\text{to inf.},\ r<e^{-\sigma}$$

$$=\frac{2}{3\pi(e^{\sigma}-r)^{\frac{5}{2}}\,k'^4}\left[2(1+k'^2)E-Kk'^2\right],\ k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}.$$

Proof:

Proceeding as in (15) the sum of the series

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\{e^{\sigma}-r-2\sinh\sigma\sin^2\theta\}^{\frac{5}{2}}}$$

$$=\frac{2}{\pi(e^{\sigma}-r)^{\frac{5}{2}}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{5}{2}}},\ k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}$$

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† See Whittaker and Watson, *Modern Analysis*, Third Edition, p. 516.

$$=\frac{2}{\pi(e^{\sigma}-r)^{\frac{5}{2}}}\int_{0}^{K}\frac{du}{\mathrm{dn}^{4}u}$$

The above integral by means of the reduction formula †

$$(n+1)\,k'^{2}\int\mathrm{nd}^{n+2}\,u\,du-n(1+k'^{2})\int\mathrm{nd}^{n}u\,du+(n-1)\int\mathrm{nd}^{n-2}\,u\,du$$
$$+k^{2}\mathrm{nd}^{n-1}\,u.\ \mathrm{cd}u\cdot\mathrm{sd}u=0$$

becomes

$$\frac{2}{3}\int_{0}^{K}\frac{1+k'^{2}}{k'^{2}}\cdot\frac{du}{\mathrm{dn}^{2}u}-\frac{K}{3k'^{2}}\,.$$

Therefore the sum of the series is equal to

$$\frac{2}{\pi(e^{\sigma}-r)^{\frac{5}{2}}}\left[\left\{\frac{2}{3}\,\frac{1+k'^{2}}{k'^{2}}\cdot\frac{E}{k'^{2}}\right\}-\frac{K}{3k'^{2}}\right]\cdot$$

$$=\frac{2}{3\pi(e^{\sigma}-r)^{\frac{5}{2}}\,k'^{4}}\,[2(1+k'^{2})E-K\,k'^{2}]$$

$$*\ (18)\quad Q_{\frac{3}{2}}(\cosh\sigma)+\frac{5}{2}\,r\,Q_{\frac{5}{2}}(\cosh\sigma)+\frac{5\cdot7}{2\cdot4}\,r^{2}Q_{\frac{7}{2}}(\cosh\sigma)+\ldots$$

$$\text{to inf., } r<e^{\sigma}.$$

$$=\frac{2}{3(e^{\sigma}-r)^{\frac{5}{2}}k^{4}}\Big[2(1+k^{2})(K-E)-k^{2}K\Big],\ k^{2}=\frac{e^{-\sigma}-r}{e^{\sigma}-r}.$$

Proof:

Proceeding as in (16) we have the sum of the series equal to

$$\frac{2}{(e^{\sigma}-r)^{\frac{5}{2}}}\int_{0}^{K}\mathrm{cd}^{4}u\,du,\quad k^{2}=\frac{e^{-\sigma}-r}{e^{\sigma}-r}\,.$$

But we have the reduction formula †

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† See *the Messenger of Mathematics*, Vol. XI, "On Elliptic functions" by J. W. L. Glaisher, pp. 120-133.

$$(n+1)k^2 \int \mathrm{cd}^{n+2} u \cdot du - n(1+k^2) \int \mathrm{cd}^n u \cdot du + (n-1) \int \mathrm{cd}^{n-2} u \cdot du$$
$$+ k'^2 \mathrm{cd}^{n-1} u \ \mathrm{sd}\, u \ \mathrm{nd}\, u = 0.$$

This gives

$$\int_0^K \mathrm{cd}^4 u du = \frac{2}{3} \frac{(1+k^2)}{k^2} \int_0^K \mathrm{cd}^2 u \, du - \frac{1}{3k^2} K$$

$$= \frac{2}{3} \frac{(1+k^2)}{k^2} \left[\frac{K-E}{k^2} \right] - \frac{1}{3k^2} K$$

$$= \frac{1}{3k^4} [2(1+k^2)(K-E) - K \cdot k^2].$$

Hence the sum of the series is equl to

$$\frac{2}{3(e^\sigma - r)^{\frac{5}{2}} \cdot k^4} [2(1+k^2)(K-E) - k^2 K]$$

*(19) $P_{\frac{5}{2}} (\cosh \sigma) + \frac{7}{2} r . P_{\frac{7}{2}} (\cosh \sigma) + \frac{7.9}{2.4} r^2 P_{\frac{9}{2}} (\cosh \sigma)$

$+ \ldots$ to inf., $r < e^{-\sigma}$

$$= \frac{2}{15\pi (e^\sigma - r)^{\frac{7}{2}} . k'^6} [8(1+k'^2)^2 E - 4(1+k'^2) k'^2 K - 9E], \ k^2 = \frac{2 \sinh \sigma}{e^\sigma - r}.$$

Proof :

Proceeding as in (17) it is easy to see that n being an odd integer

(i) $P_{\frac{n}{2}-1} (\cosh \sigma) + \frac{n}{2} \cdot r P_{\frac{n}{2}} (\cosh \sigma) + \frac{n(n+2)}{2.4} r^2 P_{\frac{n}{2}+1} (\cosh \sigma)$

$+ \ldots\ldots$ to inf., $r < e^{-\sigma}$

$$= \frac{2}{\pi (e^\sigma - r)^{\frac{n}{2}}} \left[\int_0^K \mathrm{nd}^{n-1} u . du \right], \ k^2 = \frac{2 \sinh \sigma}{e^\sigma - r},$$ which can

be evaluated by the successive use of the reduction formula†

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† Glaisher, *l.c.*, pp. 120-138.

$$k'^2(n+1)\int \mathrm{nd}^{n+2}u.du-n(1+k'^2)\int \mathrm{nd}^{n}u.du+(n-1)\int \mathrm{nd}^{n-2}u\,du$$
$$+k^2.\mathrm{nd}^{n-1}u.\mathrm{cd}u.\mathrm{sd}u=0.$$

Putting $n=7$ in (i) we have the required sum

$$=\frac{2}{\pi}\,\frac{1}{(e^{\sigma}-r)^{\frac{7}{2}}}\int_0^{K}\mathrm{nd}^6u\,du,\quad k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}.$$

$$=\frac{2}{\pi}\,\frac{1}{(e^{\sigma}-r)^{\frac{7}{2}}}\left[\frac{4(1+k'^2)}{5k'^2}\int_0^{K}\mathrm{nd}^4u.du-\frac{3}{5k'^2}\int_0^{K}\mathrm{nd}^2u.du\right]$$

$$=\frac{2}{5\pi(e^{\sigma}-r)^{\frac{7}{2}}k'^2}\left[4(1+k'^2)\left\{\frac{2}{3}\,\frac{(1+k'^2)}{k'^4}\,\mathrm{E}-\frac{\mathrm{K}}{3k'^2}\right\}-\frac{3\mathrm{E}}{k'^2}\right]$$

$$=\frac{2}{15\pi(e^{\sigma}-r)^{\frac{7}{2}}.k'^6}\,[8(1+k'^2)^2\mathrm{E}-4(1+k'^2)k'^2\mathrm{K}-9\mathrm{E}].$$

(20) $\mathrm{Q}_{\frac{5}{2}}(\cosh\sigma)+\frac{7}{2}\,r\mathrm{Q}_{\frac{7}{2}}(\cosh\sigma)+\frac{7.9}{2.4}\,r^2.\mathrm{Q}_{\frac{9}{2}}(\cosh\sigma)$

$+\ldots\ldots$to inf., $r<e^{\sigma}$

$$=\frac{2}{15(e^{\sigma}-r)^{\frac{7}{2}}k^6}\,[8(1+k^2)^2(\mathrm{K}-\mathrm{E})-4(1+k^2)\mathrm{K}k^2-9(\mathrm{K}-\mathrm{E})],$$

$$\text{where } k^2=\frac{e^{-\sigma}-r}{e^{\sigma}-r}.$$

Proof:

Proceeding as in (18) it is easy to see that n being an odd integer

(i) $\mathrm{Q}_{\frac{n}{2}-1}(\cosh\sigma)+\frac{n}{2}\,r\mathrm{Q}_{\frac{n}{2}}(\cosh\sigma)+\frac{n(n+2)}{2.4}\,r^2\mathrm{Q}_{\frac{n}{2}+1}(\cosh\sigma)$

$+\ldots\ldots$to inf., $r<e^{\sigma}$

$=\dfrac{2}{(e^{\sigma}-r)^{\frac{n}{2}}}\left[\displaystyle\int_0^{K}\mathrm{cd}^{n-1}u\,du\right]$, $k^2=\dfrac{e^{-\sigma}-r}{e^{\sigma}-r}$, which can be evaluated by the successive use of the reduction formula

$$(n+1)k^2 \int \mathrm{cd}^{n+2}u\,du - n(1+k^2)\int \mathrm{cd}^n u\,du + (n-1)\int \mathrm{cd}^{n-2}u\,du$$
$$+k'^2\mathrm{cd}^{n-1}u\,.\mathrm{sd}u.\,\mathrm{nd}u = 0$$

Putting $n=7$ in (i) we have the required sum

$$= \frac{2}{(e^\sigma - r)^{\frac{7}{2}}}\int_0^K \mathrm{cd}^6 u.du$$

$$= \frac{2}{5(e^\sigma - r)^{\frac{7}{2}}}\left[\frac{4(1+k^2)}{k^2}\int_0^K \mathrm{cd}^4 u du - \frac{3}{k^2}.\int_0^K \mathrm{cd}^2 u.du\right]$$

$$= \frac{2}{5(e^\sigma - r)^{\frac{7}{2}}}.\left[\frac{4(1+k^2)}{k^2}\left\{\frac{1}{3k^4}\,2(1+k^2)(\mathrm{K}-\mathrm{E}) - \frac{\mathrm{K}}{3k^2}\right\}\right.$$
$$\left. -\frac{3}{k^2}\left\{\frac{(\mathrm{K}-\mathrm{E})}{k^2}\right\}\right]$$

$$= \frac{2}{15(e^\sigma - r)^{\frac{7}{2}}.k^6}.\left[8(1+k^2)^2(\mathrm{K}-\mathrm{E}) - 4(1+k^2)\mathrm{K}k^2 - 9(\mathrm{K}-\mathrm{E})\right].$$

*(21) $\mathrm{P}_{-\frac{1}{4}}(\cosh\sigma) + \frac{3}{4}r\mathrm{P}_{\frac{3}{4}}(\cosh\sigma) + \frac{3.7}{4.8}r^2\mathrm{P}_{\frac{7}{4}}(\cosh\sigma)$

$+\ldots\ldots$to inf., $r < e^{-\sigma}$

$= \dfrac{2\mathrm{K}}{\pi(e^\sigma - r)^{\frac{3}{4}}\sqrt{k'}\,\sqrt{2(1+k')}}$, where the modulus corresponding to K is $\dfrac{1-\sqrt{k'}}{\sqrt{2(1+k')}}$ and $k^2 = 1 - k'^2 = \dfrac{2\sinh\sigma}{e^\sigma - r}$.

Proof :

Proceeding as in (15) the required sum

$$= \frac{1}{\pi}\int_0^\pi d\phi\left\{\sum_{m=0}^{\infty}\frac{r^m}{\mathrm{D}^{m+\frac{3}{4}}}.\frac{3.7.11\ldots 4m-1}{4.8.12\ldots 4m}\right\}$$

$$=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{(D-r)^{\frac{3}{2}}}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(e^{\sigma}-r-2\sinh\sigma\sin^2\theta)^{\frac{3}{2}}}\quad\left(\text{putting }\frac{\phi}{2}=\theta\right)$$

$$=\frac{2}{\pi(e^{\sigma}-r)^{\frac{3}{2}}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}}\quad,\left(k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}\right)$$

$$=\frac{4K}{\pi(e^{\sigma}-r)^{\frac{3}{2}}\sqrt{k'}\sqrt{2(1+k')}}$$, modulus corresponding to K being $\frac{1-\sqrt{k'}}{\sqrt{2(1+k')}}$. (See Greenhill's *Elliptic Functions*, p. 165.)

*(22) $\frac{r^2}{2}.\,P_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\cdot\frac{r^3}{3}.\,P_{\frac{1}{2}}(\cosh\sigma)+\frac{1.3}{2.4}\cdot\frac{r^4}{4}\,P_{\frac{3}{2}}(\cosh\sigma)$

+......to inf., is easily summed for $r<e^{-\sigma}$.

Proof :

Multiplying each term of

$$P_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,.\,P_{\frac{1}{2}}(\cosh\sigma).\,r+\frac{1.3}{2.4}\,r^2P_{\frac{3}{2}}(\cosh\sigma+\ldots\ldots\text{to inf.}$$

by r and integrating the series with respect to r from $r=0$ to $r=r$ we have the required sum

$$=\frac{1}{\pi}\int_0^{\pi}d\phi.\int_0^{r}\frac{rdr}{\sqrt{D-r}}$$ (D standing for $\cosh\sigma+\sinh\sigma.\cos\phi$.

Now

$$\int_0^r\frac{rdr}{\sqrt{D-r}}=\int_0^r\frac{Ddr}{\sqrt{D-r}}-\int_0^r\sqrt{D-r}\;dr$$

$$=-2\,D\left[\sqrt{D-r}\right]_0^r+\frac{2}{3}\cdot\left[(D-r)^{\frac{3}{2}}\right]_0^r$$

$$=\frac{2}{3}\,(D-r)^{\frac{3}{2}}-2D\sqrt{D-r}+\frac{4}{3}\,D^{\frac{3}{2}}$$

Therefore the sum

$$=\frac{2}{\pi}\int_0^{\pi}\frac{1}{3}(D-r)^{\frac{3}{2}}d\phi-\frac{2}{\pi}\int_0^{\pi}D\sqrt{D-r}\,d\phi+\frac{4}{3}\frac{1}{\pi}\int_0^{\pi}D^{\frac{3}{2}}d\phi$$

$$=\frac{4}{3\pi}\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\theta)^{\frac{3}{2}}(e^{\sigma}-r)^{\frac{3}{2}}d\theta+\frac{8}{3\pi}$$

$$\times\int_0^{\frac{\pi}{2}}(1-k_1^2\sin^2\theta)^{\frac{3}{2}}(e^{\sigma})^{\frac{3}{2}}d\theta$$

$$-\frac{4}{\pi}\int_0^{\frac{\pi}{2}}(e^{\sigma}-r)^{\frac{1}{2}}e^{\sigma}(1-k_1^2\sin^2\theta)(1-k^2\sin^2\theta)^{\frac{1}{2}}d\theta$$

(Putting $\phi=2\theta$)

where $k^2=\dfrac{2\sinh\sigma}{e^{\sigma}-r}$ and $k_1^2=\dfrac{2\sinh\sigma}{e^{\sigma}}$.

$$=\frac{4}{3\pi}(e^{\sigma}-r)^{\frac{3}{2}}\int_0^{F(k)}\mathrm{dn}^4u\cdot du+\frac{8}{3\pi}(e^{\sigma})^{\frac{3}{2}}\int_0^{F(k_1)}\mathrm{dn}^4v\cdot dv$$

$$-\frac{4}{\pi}(e^{\sigma}-r)^{\frac{1}{2}}e^{\sigma}.\int_0^{F(k)}(1-k_1^2\sin^2u)\,\mathrm{dn}^2u.\,du$$

where $\mathrm{dn}^2u=1-k^2\sin^2\theta$ and $\mathrm{dn}^2v=1-k_1^2\sin^2\theta$

$$=\frac{4}{9\pi}(e^{\sigma}-r)^{\frac{3}{2}}\{2(1+k'^2)\,E(k)-k'^2F(k)\}$$

$$+\frac{8}{9\pi}(e^{\sigma})^{\frac{3}{2}}\{2(1+k_1'^2)E(k_1)-k_1'^2F(k_1)\}$$

$$-\frac{4}{\pi}e^{\sigma}(e^{\sigma}-r)^{\frac{1}{2}}\int_0^{F(k)}\{1+k_1^2k^2\mathrm{sn}^4u-(k^2+k_1^2)\mathrm{sn}^2u\}du$$

where $k'^2=1-k^2$ and $k_1'^2=1-k_1^2$

where the modulus corresponding to K and E is $\dfrac{1-\sqrt{k'}}{\sqrt{2(1+k')}}=l$ (say).

The required sum is, therefore,

$$=\frac{2}{\pi}\,(e^{\sigma}-r)^{\frac{1}{4}}\left[\frac{\sqrt{k'}}{\sqrt{2(1+k')}}\left\{-2K\right\}+\frac{\sqrt{2(1+k')}}{2}\left\{\beta\left(\frac{2(K-E)}{l^2}\right)\right.\right.$$

$$\left.\left.+\frac{2E-2(1-l^2)K}{l^2}\right\}\right]$$

*(25) $P_{-\frac{7}{4}}(\cosh\sigma)-\frac{3}{4}\,rP_{-\frac{3}{4}}(\cosh\sigma)-\frac{3.1}{4.8}\,r^2P_{\frac{1}{4}}(\cosh\sigma)$

$$-\frac{3.1.5}{4.8.12}\,r^3P_{\frac{5}{4}}(\cosh\sigma)-\ldots\quad\text{to inf.},\ r<e^{-\sigma}$$

can be easily summed up.

Proof:

[illegible] the required sum is easily found to be

Now proceeding as for $\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-k^2 \sin^2\theta)^{\frac{1}{4}}}$ we find that

$$\int_0 (1-k^2 \sin^2\theta)^{\frac{1}{4}} d\theta = \frac{1}{2} \frac{1+h'}{\sqrt{1+k'}} \left[\int \frac{y\,dy}{\sqrt{(1-y)(1+y)(y-\beta)}} \right.$$

$$\left. + \int \frac{y\,dy}{\sqrt{(1-y)(1+y)(y-\beta')}} \right] + \frac{1}{2} \frac{\sqrt{k'}}{\sqrt{1+k'}} \left[\int \frac{dy}{\sqrt{(1-y)(1+y)(y-\beta')}} \right.$$

$$\left. - \int \frac{dy}{\sqrt{(1-y)(1+y)(y-\beta)}} \right],$$

where $y = \frac{\iota^4 + k'}{(1+k')\iota^2}$, $\iota^2 = \sqrt{1-k^2 \sin^2\theta}$, $k'^2 = 1-k^2$,

$$\beta = \frac{2\sqrt{k'}}{1+k'} \text{ and } \beta' = \frac{-2\sqrt{k'}}{1+k'}.$$

Now making the substitution

$$\sqrt{2}u = \int^1 \frac{dy}{[illegible]}$$

$$= \frac{2}{\pi} (e^{\sigma} - r)^{\frac{3}{4}} \int_{0}^{\frac{\pi}{2}} (1 - k^2 \sin^2\theta)^{\frac{3}{4}} d\theta, \quad k^2 = \frac{2 \sinh \sigma}{e^{\sigma} - r}$$

$$= \frac{2}{\pi} (e^{\sigma} - r)^{\frac{3}{4}} \left[\frac{1}{2} \frac{\sqrt{k'}(1+k')}{\sqrt{1+k'}} \left\{ \int_{y}^{1} \frac{y\,dy}{\sqrt{(1-y)(1+y)(y-\beta')}} \right.\right.$$

$$\left. - \int_{y}^{1} \frac{y\,dy}{\sqrt{(1-y)(1+y)(y-\beta)}} \right\} - \frac{2k'\mathrm{K}}{\sqrt{2(1+k')}} + \frac{1}{2} \frac{(1+k')^2}{\sqrt{1+k'}}$$

$$\times \left[\int_{y}^{1} \frac{y^2\,dy}{\sqrt{(1-y)(1+y)(y-\beta')}} + \int_{y}^{1} \frac{y^2\,dy}{\sqrt{(1-y)(1+y)(y-\beta)}} \right.$$

where $k'^2 = 1 - k^2$ and the modulus corresponding to k is $l = \dfrac{1 - \sqrt{k'}}{\sqrt{2(1+k')}}$ after the method used in (24).

$$[illegible] \overline{\sqrt{(1-y)(1+y)(y-\beta)}} = \sqrt{2}\ \mathrm{cn}^{-1} \sqrt{\frac{y-\beta}{1-\beta}}$$

and a similar substitution

$$\sqrt{2}\, v = \int_y^1 \frac{dy}{\sqrt{(1-y)(1+y)(y-\beta')}} = \sqrt{2}\ \mathrm{cn}^{-1} \sqrt{\frac{y-\beta'}{1-\beta'}}$$

we find that

$$\int_0^{\frac{\pi}{2}} (1-k^2 \ \sin^2\theta)^{\frac{1}{2}} d\theta$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2k'}}{\sqrt{1+k'}} \left[-2\mathrm{K} \right] + \frac{\sqrt{2(1+k')}}{2} \left[\beta \left\{ \frac{2(\mathrm{K}-\mathrm{E})}{l^2} \right\} \right.$$

$$\left. + \frac{2\mathrm{E}-2(1-l^2)\mathrm{K}}{l^2} \right]$$

---

* Greenhill, *Elliptic Functions*, p. 164.

$$=\frac{2}{\pi}\left(e^{\sigma}-r\right)^{\frac{1}{4}}\left[\frac{\sqrt{2}}{2}\,\frac{(1+k')^{2}}{\sqrt{1+k'}}\left\{\beta^{2}\cdot 2\mathrm{K}+2\beta(1-\beta)\cdot\frac{2\mathrm{E}-2(1-l^{2})\mathrm{K}}{l^{2}}\right.\right.$$

$$\left.+\,(1-\beta)^{2}\left(\frac{2(2l^{2}-1)}{3l^{4}}\{2\mathrm{E}-2\mathrm{K}(1-l^{2})\}+\frac{1-l^{2}}{3l^{2}}\,2\mathrm{K}\right)\right\}$$

$$-\frac{2\mathrm{K}k'}{\sqrt{2(1+k')}}-\frac{1}{2}\,\frac{\sqrt{k'}(1+k')}{\sqrt{1+k'}}\left\{\beta\,\frac{2\mathrm{K}-2\mathrm{E}}{l^{2}}\right.$$

$$\left.\left.+\,\frac{2\mathrm{E}-2(1-l^{2})\mathrm{K}}{l^{2}}\right\}\right].$$

modulus corresponding to K and E being l as in (24).

* (26)

$$\frac{r^{2}}{2}\,\mathrm{P}_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\cdot\frac{r^{3}}{2\cdot 3}\,\mathrm{P}_{\frac{1}{2}}(\cosh\sigma)+\frac{1\cdot 3}{2\cdot 4}\,\frac{r^{4}}{3\cdot 4}\,\mathrm{P}_{\frac{3}{2}}(\cosh\sigma)+\ldots\text{to inf.}$$

$$r<e^{-\sigma},$$

$=\Phi$, a function expressible in terms of elliptic functions.

Proof :

Integrating n times with respect to r between zero and r the series

$$\mathrm{P}_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,r\cdot\mathrm{P}_{\frac{1}{2}}(\cosh\sigma)+\frac{1\cdot 3}{2\cdot 4}\,r^{2}\mathrm{P}_{\frac{3}{2}}(\cosh\sigma)+\ldots\text{to inf.},\; r<e^{-\sigma}$$

$$=\int_{0}^{\pi}\frac{d\phi}{\sqrt{\cosh\sigma-r+\sinh\sigma\cdot\cos\phi}}\text{, we see that}$$

(i) $$\frac{r^{n}}{1\cdot 2\cdot 3\ldots n}\cdot\mathrm{P}_{-\frac{1}{2}}(\cosh\sigma)+\frac{1}{2}\,\frac{r^{n+1}}{2\cdot 3\ldots(n+1)}\cdot\mathrm{P}_{\frac{1}{2}}(\cosh\sigma)$$

$$+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{r^{n+2}}{3\cdot 4\ldots(n+2)}\,\mathrm{P}_{\frac{3}{2}}(\cosh\sigma)+\ldots\ \text{to inf.},\; r<e^{-\sigma}$$

$$=\frac{1}{\pi}\Bigg[\frac{2^2\cdot r^{n-1}}{(n-1)!}\sqrt{e^{\sigma}}\,\mathrm{E}(k_1)-\frac{2^3}{3}\cdot\frac{r^{n-2}}{(n-2)!}(e^{\sigma})^{\frac{3}{2}}\frac{1}{3}\Big\{2(1+k_1^2)\,\mathrm{E}(k_1)$$

$$-k_1'^2\mathrm{F}(k_1)\Big\}+\frac{2^4\cdot 2!\cdot r^{n-3}}{3\cdot 5\cdot (n-3)!}(e^{\sigma})^{\frac{5}{2}}\Big\{\frac{4}{15}(1+k_1'^2)\Big(2(1+k_1'^2)\,\mathrm{E}(k_1)$$

$$-k_1'^2\,\mathrm{F}(k_1)\Big)-\frac{3}{5}k_1'^2\,\mathrm{E}(k_1)\Big\}$$

$$+\dots(-1)^{n-1}\frac{2^{n+1}}{3\cdot 5\dots(2n-1)}(e^{\sigma})^{n-\frac{1}{2}}\int_0^{\mathrm{F}(k_1)}\mathrm{dn}^{2n}v\,dv$$

$$+(-1)^{n}\frac{2^{n+1}}{3\cdot 5\dots(2n-1)}(e^{\sigma}-r)^{n-\frac{1}{2}}\int_0^{\mathrm{F}(k)}\mathrm{dn}^{2n}u\,du\Bigg]$$

where $\mathrm{dn}^2v=1-k_1^2\sin^2\theta$, $\mathrm{dn}^2u=1-k^2\sin^2\theta$,

$$k_1^2=\frac{2\sinh\sigma}{e^{\sigma}}\;,\quad k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}\;,\quad k_1'^2=1-k_1^2\text{ and }k'^2=1-k^2,$$

the last two integrals being evaluated by the successive use of the reduction formula *

$$\int\mathrm{dn}^{2n}u\,du=\frac{2(n-1)(1+k'^2)}{2n-1}\int\mathrm{dn}^{2n-2}u\cdot du$$

$$-\frac{(2n-3)k'^2}{2n-1}\int\mathrm{dn}^{2n-4}u\,du+\frac{1}{2n-1}k^2\mathrm{dn}^{2n-3}u\cdot\mathrm{sn}\,u\cdot\mathrm{cn}\,u=0.$$

Putting $n=2$ in (i) we have the required sum $=\Phi$

$$=\frac{4}{\pi}r\sqrt{e^{\sigma}}\,\mathrm{E}(k_1)+\frac{8}{9\pi}(e^{\sigma}-r)^{\frac{3}{2}}\Big\{2(1+k'^2)\,\mathrm{E}(k)-k'^2\,\mathrm{F}(k)\Big\}$$

$$-\frac{8}{9\pi}(e^{\sigma})^{\frac{3}{2}}\Big\{2(1+k_1'^2)\mathrm{E}(k_1)-k_1'^2\mathrm{F}(k_1)\Big\}.$$

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* J. W. L. Glaisher, *l.c.*, pp. 120-138.

A note on my first paper. †

I take this opportunity to make some corrections and additions in y first paper.

Errata.

24. Line 7 from the top, *read* | a | or | r | *for* a or r.
25. For the sum of the series (6) *read* in the first term,

$$\frac{1}{6}\left\{\frac{5x^3+9x^2+3x-1}{2}\right\}\log\frac{1+x+\sqrt{2(1+x)}}{1+x},$$

$$\frac{1}{6}\left\{\frac{5x^3+9x^2+3x-1}{2}\right\}\log\frac{1+x+\sqrt{2(1-x)}}{1+x}.$$

7. In the series (18), *for* K having the same meaning as in (17) the modulus corresponding to K, the complete elliptic integral he first kind, being $\sin\frac{\theta}{4}$.

Additions.

$$\sum_{=0}^{\infty}\frac{(-1)^{n-1}\,P_{2n+1}(x)}{2(n+3)}=\left\{\frac{2}{3}\cdot\frac{7}{4}x-\frac{5.7}{2.4}x^3+\frac{3.3}{4.2}x\right\}$$

$$\frac{7}{4.3}x^{\frac{1}{2}}+\frac{\sqrt{x}}{2}\left\{\frac{5.7}{3.4}x^2-\frac{3}{4}\right\}+\sqrt{x}\left\{-\frac{7.2}{4.3}x+\frac{5.7}{2.4}x^3\right.$$

$$\left\{\frac{1}{2}\log(1+x)+\log\frac{1-\sqrt{x}}{1-x}\right\}\left\{\frac{7x}{4}\left(\frac{5}{2}x^3-\frac{1.5}{2.3}x-\frac{2}{3}x\right)\right.$$

$$\left.-\frac{3}{4}\left(\frac{3x^2}{2}-\frac{1}{2}\right)\right\}.$$

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ation of infinite series of Legendre's polynomials," *Bulletin* *Society*, Vol. XXIII, No 1, pp. 28-44.

$$* \text{(ii)} \quad \sum_{n=0}^{\infty} \frac{(-1)^n \cdot P_{2n}(x)}{2n+5} = -\frac{\sqrt{x}}{4} + \frac{7}{4.3} x^{\frac{3}{2}} + \frac{\sqrt{x}}{2} \left\{ \frac{5.7}{4.3} x^2 - \frac{3}{4} \right\}$$

$$-\sqrt{x} \left\{ -\frac{7.2}{4.3} x + \frac{5}{2} \cdot \frac{7}{4} \cdot x^3 - \frac{3.3}{2.4} x \right\}$$

$$+ \left\{ \frac{7x}{4} \left(\frac{5x^3}{2} - \frac{1.5}{2.3} x - \frac{2}{3} x \right) - \frac{3}{4} \left(\frac{3x^2}{2} - \frac{1}{2} \right) \right\} \tan^{-1}$$

These two series can be obtained by putting $a=i$ in the series (7)† equating the real and imaginary parts.

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† *L.c.*, p. 25.

Bull. Cal. Math. Soc., Vol. XVIII, No. 3, (1931).

MENDED
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